

Signals and Systems

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Course Overview

- Tentative Schedule

#	Topic	Assignment
1	Introduction	
2	Representing Signals	
3	Continuous-Time System	
4	Complex Numbers	
5	Fourier Series	
6	Fourier Transform	
7	Fourier Analysis of Discrete-Time System	DFT (self-motivated)

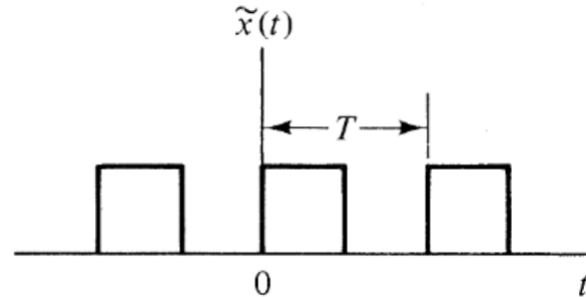
6. Fourier Transform

Fourier Transform

- Continuous Time Fourier Transform
 - Main Concept : Periodic $\rightarrow$ Aperiodic (Generalization)
 - Occupies a continuum of frequencies

$$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} c_n \exp[jn\omega_0 t]$$

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} \tilde{x}(t) \exp[-jn\omega_0 t] dt$$



Fourier Transform

- Continuous Time Fourier Transform

- $x(t)$ is an aperiodic function.
- Define x_T as follows:

$$x_T(t) = \begin{cases} x(t), & -\frac{T}{2} \leq t < \frac{T}{2} \\ 0, & \text{otherwise} \end{cases}$$

- Define $\tilde{x}(t)$

- For $-\frac{T}{2} < t < \frac{T}{2}$, $\tilde{x}(t) = x(t)$ and $\tilde{x}(t) = \tilde{x}(t + T)$

$$\lim_{T \rightarrow \infty} x_T(t) = x(t).$$

$$\lim_{T \rightarrow \infty} \tilde{x}(t) = x(t).$$

$$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} c_n \exp[jn\omega_0 t]$$

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} \tilde{x}(t) \exp[-jn\omega_0 t] dt$$

Fourier Transform

- Continuous Time Fourier Transform

- Since $\tilde{x}(t)$ is a periodic function, the Fourier Series will be

$$\tilde{x}(t) = \sum_{n=-\infty}^{\infty} c_n \exp[jn\omega_0 t]$$

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} \tilde{x}(t) \exp[-jn\omega_0 t] dt$$

- The above can be rewritten as

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} \tilde{x}(t) e^{-jn\omega_0 t} dt = \frac{1}{T} \int_{-T/2}^{T/2} x_T(t) e^{-jn\omega_0 t} dt$$

- Consider that $x_T(t) = 0$ for $|t| > T/2$,

$$c_n = \frac{1}{T} \int_{-\infty}^{\infty} x_T(t) e^{-jn\omega_0 t} dt$$

Fourier Transform

- Continuous Time Fourier Transform
 - Substituting to Fourier series definition,

$$\begin{aligned}\tilde{x}(t) &= \sum_{n=-\infty}^{\infty} \left[\frac{1}{T} \int_{-\infty}^{\infty} x_T(t) e^{-jn\omega_0 t} dt \right] e^{jn\omega_0 t} \\ &= \sum_{n=-\infty}^{\infty} \left[\frac{\omega_0}{2\pi} \int_{-\infty}^{\infty} x_T(t) e^{-jn\omega_0 t} dt \right] e^{jn\omega_0 t} \\ &= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_T(t) e^{-jn\omega_0 t} dt \right] e^{jn\omega_0 t} \cdot \omega_0\end{aligned}$$

Fourier Transform

- Continuous Time Fourier Transform
 - Substituting to Fourier series definition,
 - As $T \rightarrow \infty$, $\omega_0 \rightarrow 0$. $k\omega_0$ becomes cont. variable
 - As $T \rightarrow \infty$, $x_T(t) \rightarrow x(t)$

$$\tilde{x}(t) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_T(t) e^{-jn\omega_0 t} dt \right] e^{jn\omega_0 t} \cdot \omega_0$$

$$x(t) = \lim_{T \rightarrow \infty} \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \boxed{\left[\int_{-\infty}^{\infty} x_T(t) e^{-jn\omega_0 t} dt \right]} e^{jn\omega_0 t} \cdot \omega_0$$

$X(\omega)$

Fourier Transform

- Continuous Time Fourier Transform

- The above can be rewritten as

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} \tilde{x}(t) e^{-jn\omega_0 t} dt = \frac{1}{T} \int_{-T/2}^{T/2} x_T(t) e^{-jn\omega_0 t} dt$$

Fourier Transform

- Continuous Time Fourier Transform
 - The variable of each functions are continuous!
 - Fourier Series : for discrete frequencies.

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \exp[j\omega t] d\omega$$

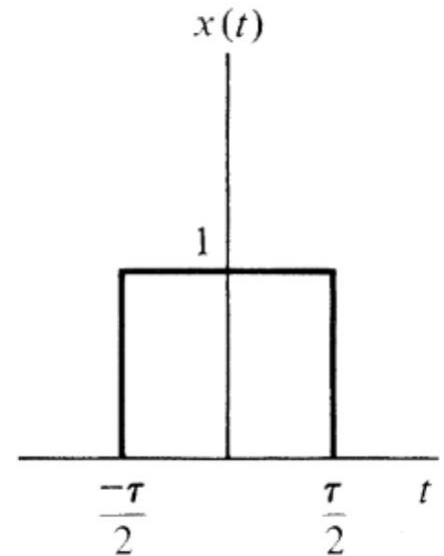
$$X(\omega) = \int_{-\infty}^{\infty} x(t) \exp[-j\omega t] dt$$

- The Fourier Transform is **complex function, with ω**

$$X(\omega) = |X(\omega)| \exp[j\phi(\omega)]$$

Fourier Transform

- Example
 - $x(t) = \text{rect}\left(\frac{t}{\tau}\right)$

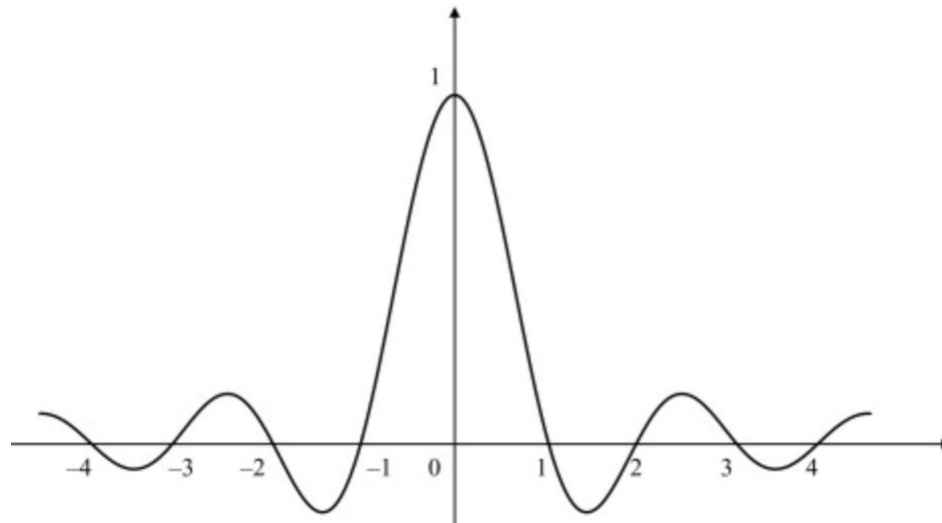


Fourier Transform

- Example

- $x(t) = \text{rect}\left(\frac{t}{\tau}\right) \rightarrow X(\omega) = \tau \text{sinc} \frac{\omega\tau}{2\pi}$

- Bandwidth (Null-to-null) : $2\pi/\tau$ rad/s, $1/\tau$ Hz
 - Mainlobe
 - Sidelobe



Fourier Transform

- Examples with Delta Function

- $x(t) = \delta(t)$

- $X(\omega) = \delta(\omega)$

- $X(\omega) = \delta(\omega - \omega_0)$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \exp[j\omega t] d\omega$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t) \exp[-j\omega t] dt$$

Fourier Transform

- Examples with Sinusoidal Function

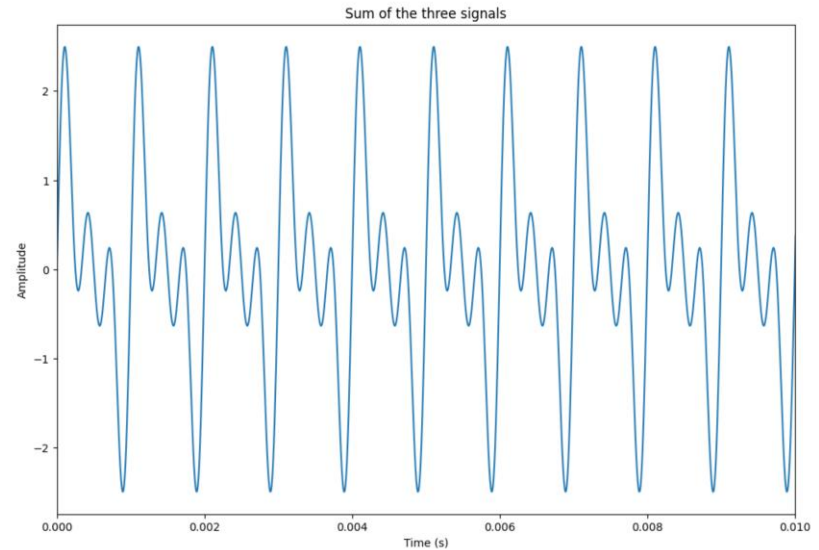
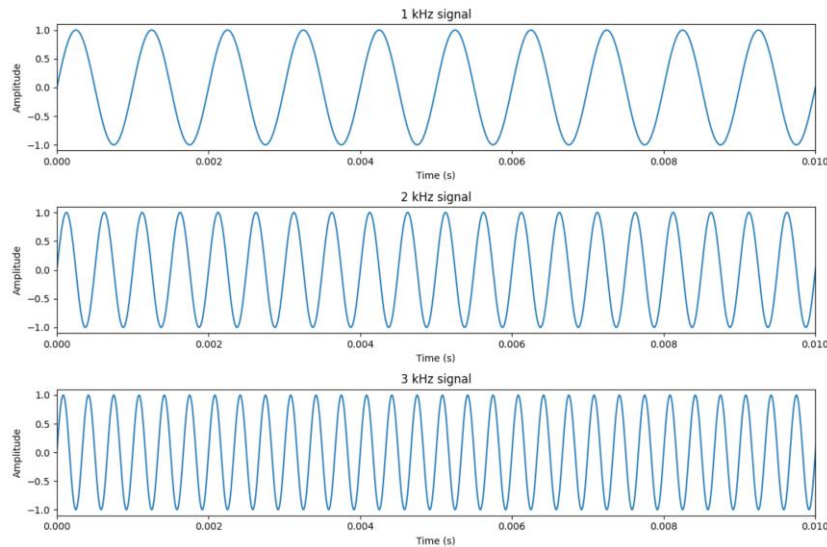
- $x(t) = \cos \omega_0 t = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$

- $x(t) = \sin \omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$

Fourier Transform

- Examples
 - Sum of periodic signals in time domain.

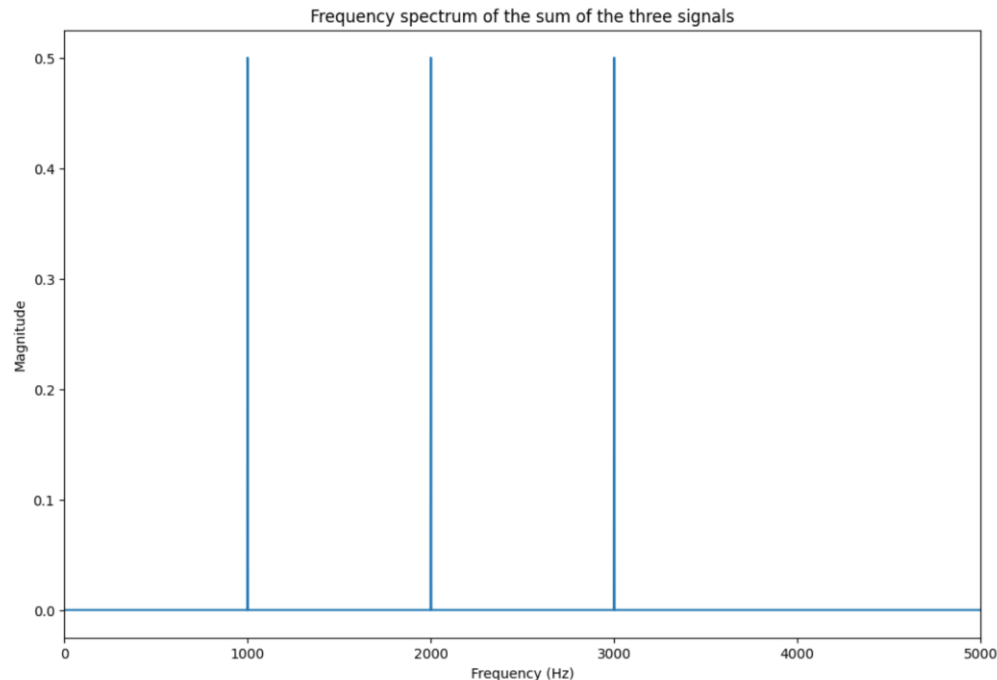
$$x(t) = \sum_{n=-\infty}^{\infty} c_n \exp[jn\omega_0 t]$$



Fourier Transform

- Examples
 - A sum of periodic signals can be represented individually.

$$x(t) = \sum_{n=-\infty}^{\infty} c_n \exp[jn\omega_0 t]$$



Fourier Transform

- Examples

- What if $x(t)$ is sum of impulse functions,
- Let's see the result with Python

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \quad c_n = \frac{1}{T} \int_{\langle T \rangle} x(t) \exp\left[-\frac{j2\pi nt}{T}\right] dt = \frac{1}{T}$$

$$x(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T} \exp\left[\frac{j2\pi nt}{T}\right] \quad \longrightarrow \quad X(\omega) = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi n}{T}\right)$$

$$\sum_{n=-\infty}^{\infty} \delta(t - nt_0) \quad \leftrightarrow \quad \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$$

Fourier Transform

- Fourier Transform Pairs

$x(t)$	$x(\omega)$
1. 1	$2\pi \delta(\omega)$
2. $u(t)$	$\pi \delta(\omega) + \frac{1}{j\omega}$
3. $\delta(t)$	1
4. $\delta(t - t_0)$	$\exp[-j\omega t_0]$
5. $\text{rect}(t/\tau)$	$\tau \text{sinc} \frac{\omega\tau}{2\pi} = \frac{2 \sin \omega\tau/2}{\omega}$
6. $\frac{\omega_B}{\pi} \text{sinc} \frac{\omega_B t}{\pi} = \frac{\sin \omega_B t}{\pi t}$	$\text{rect}(\omega/2\omega_B)$
7. $\text{sgn } t$	$\frac{2}{j\omega}$
8. $\exp[j\omega_0 t]$	$2\pi \delta(\omega - \omega_0)$
9. $\sum_{n=-\infty}^{\infty} a_n \exp[jn\omega_0 t]$	$2\pi \sum_{n=-\infty}^{\infty} a_n \delta(\omega - n\omega_0)$
10. $\cos \omega_0 t$	$\pi [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
11. $\sin \omega_0 t$	$\frac{\pi}{j} [\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$
12. $(\cos \omega_0 t)u(t)$	$\frac{\pi}{2} [\delta(\omega - \omega_0) + \delta(\omega + \omega_0)] + \frac{j\omega}{\omega_0^2 - \omega^2}$

Fourier Transform

- Properties of Fourier Transform
 - Same as Fourier Series!
 - Linearity
 - **Symmetry**
 - Time Shift
 - Time Scaling
 - **Differentiation**
 - **Convolution**
 - Duality
 - Modulation or Frequency Shift

Fourier Transform

- Properties of Fourier Transform
 - **Symmetry**

$$X(-\omega) = X^*(\omega)$$

$$|X(\omega)| = |X(-\omega)|, \phi(\omega) = -\phi(-\omega)$$

Fourier Transform

- Properties of Fourier Transform
 - **Differentiation**

$$\frac{dx(t)}{dt} \leftrightarrow j\omega X(\omega)$$

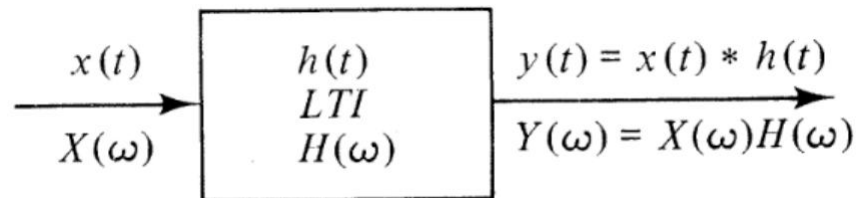
$$\frac{d^n x(t)}{dt^n} \leftrightarrow (j\omega)^n X(\omega)$$

Fourier Transform

- Properties of Fourier Transform
 - **Convolution**

$$x(t) * h(t) \leftrightarrow X(\omega)H(\omega)$$

$$x(t)h(t) \leftrightarrow \frac{1}{2\pi} X(\omega) * H(\omega)$$



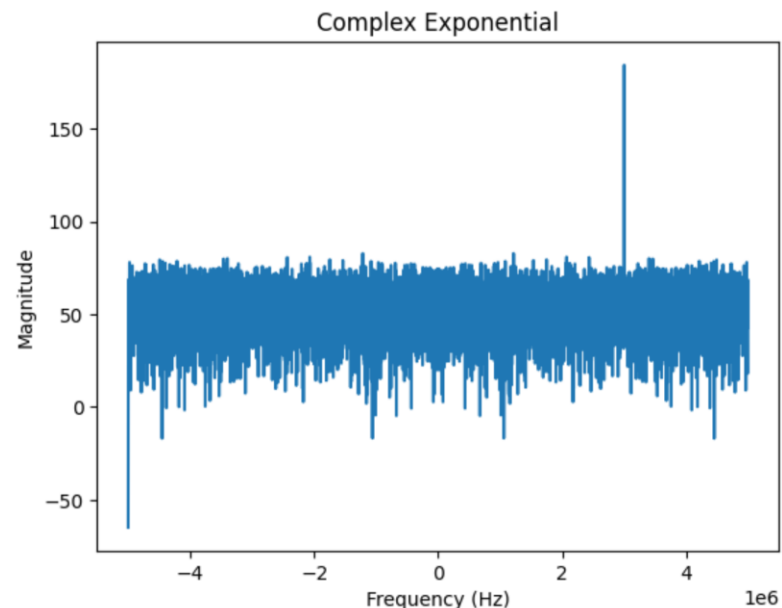
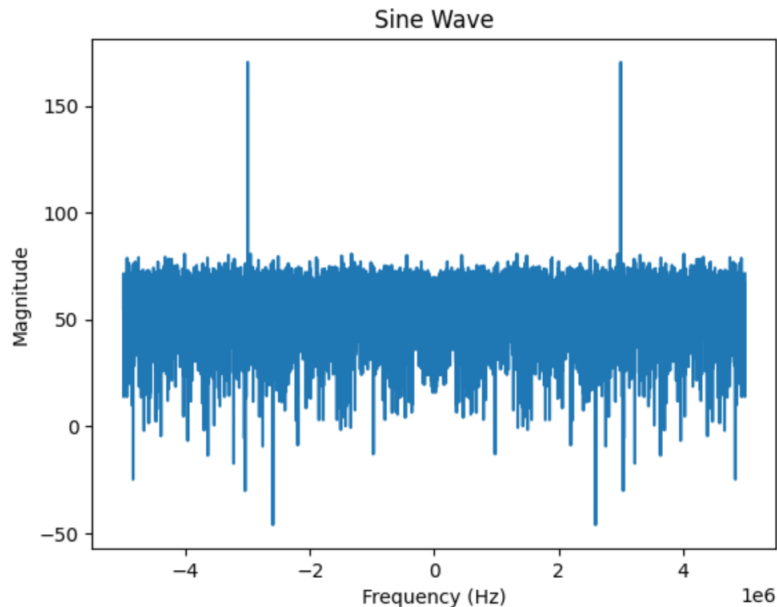
- One of the most useful in comm., control, and signal proc.

Fourier Transform

- Properties of Fourier Transform
 - **Symmetry** (only when the function x is real)

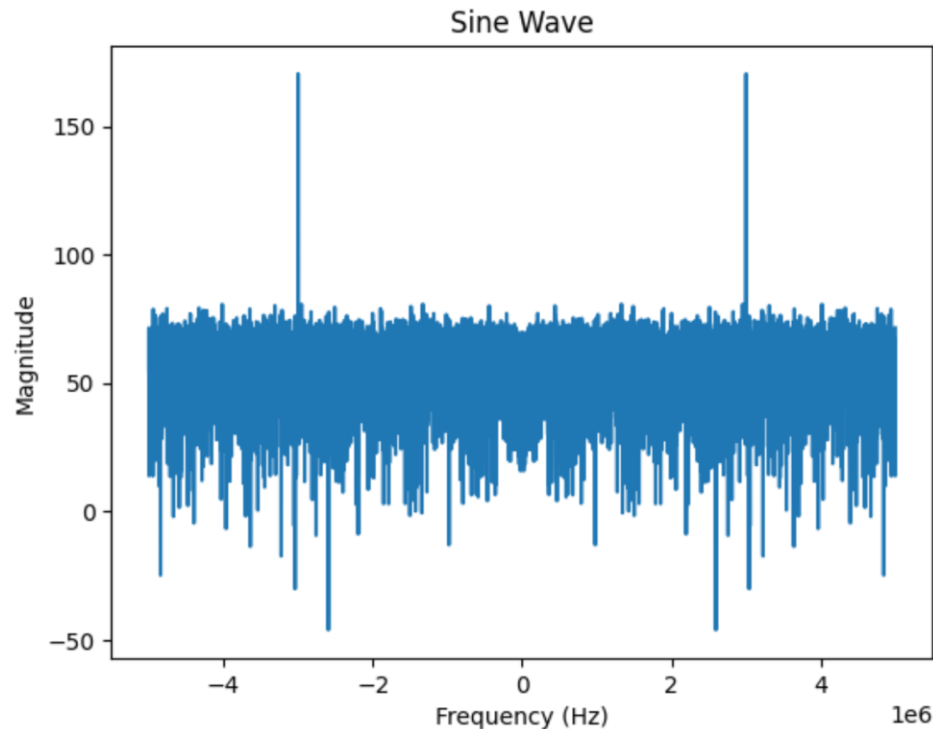
$$X(-\omega) = X^*(\omega)$$

$$|X(\omega)| = |X(-\omega)|, \phi(\omega) = -\phi(-\omega)$$



Fourier Transform

- Properties of Fourier Transform
 - When the function x is real
 - The negative frequency is ignored. (because it's symmetric)

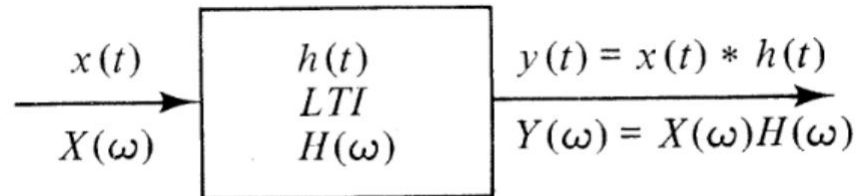


Fourier Transform

- Properties of Fourier Transform
 - **Convolution**

$$x(t) * h(t) \leftrightarrow X(\omega)H(\omega)$$

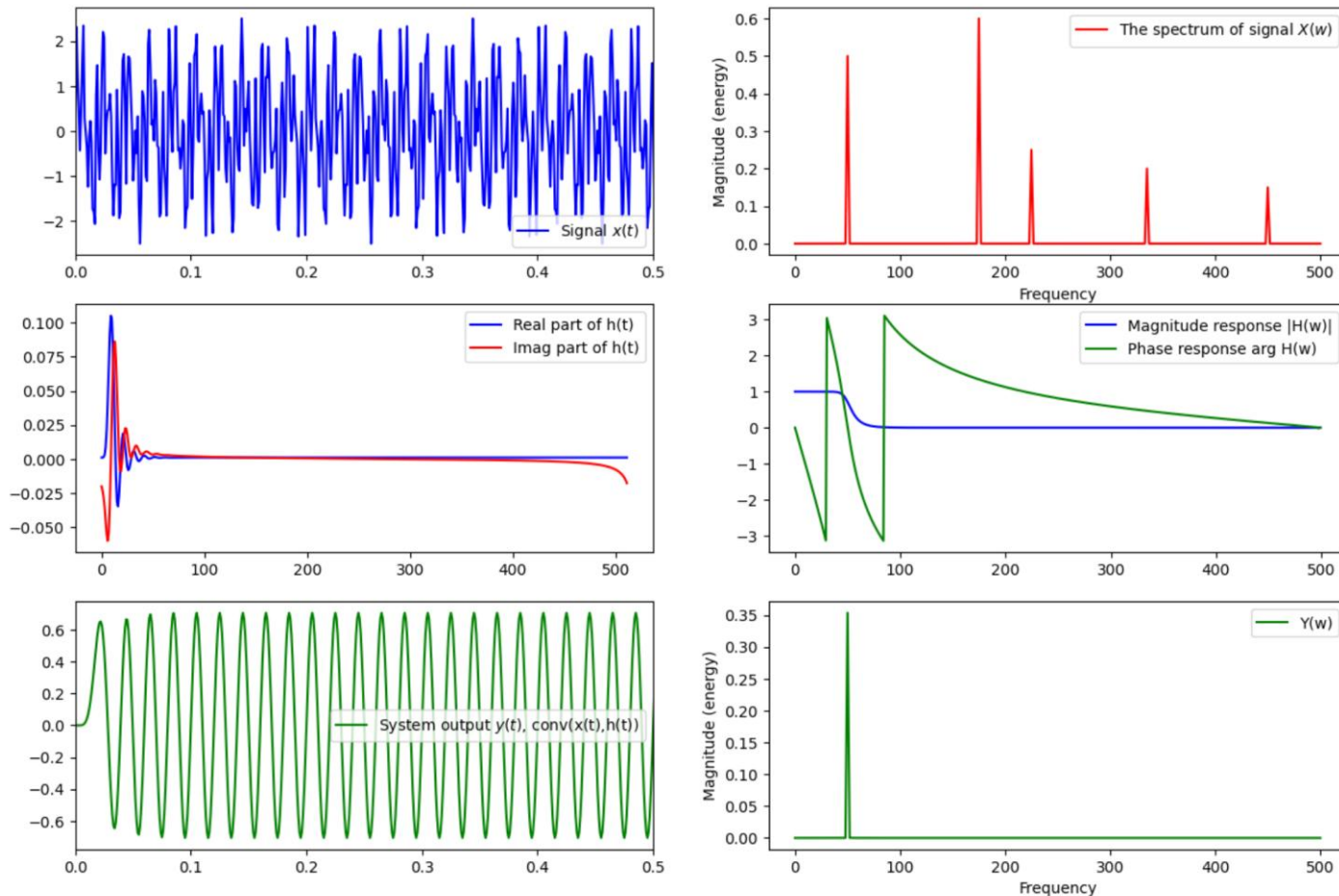
$$x(t)h(t) \leftrightarrow \frac{1}{2\pi} X(\omega) * H(\omega)$$



- The $H(\omega)$ is called **frequency response**.
- An LTI system is completely characterized by $H(\omega)$

Fourier Transform

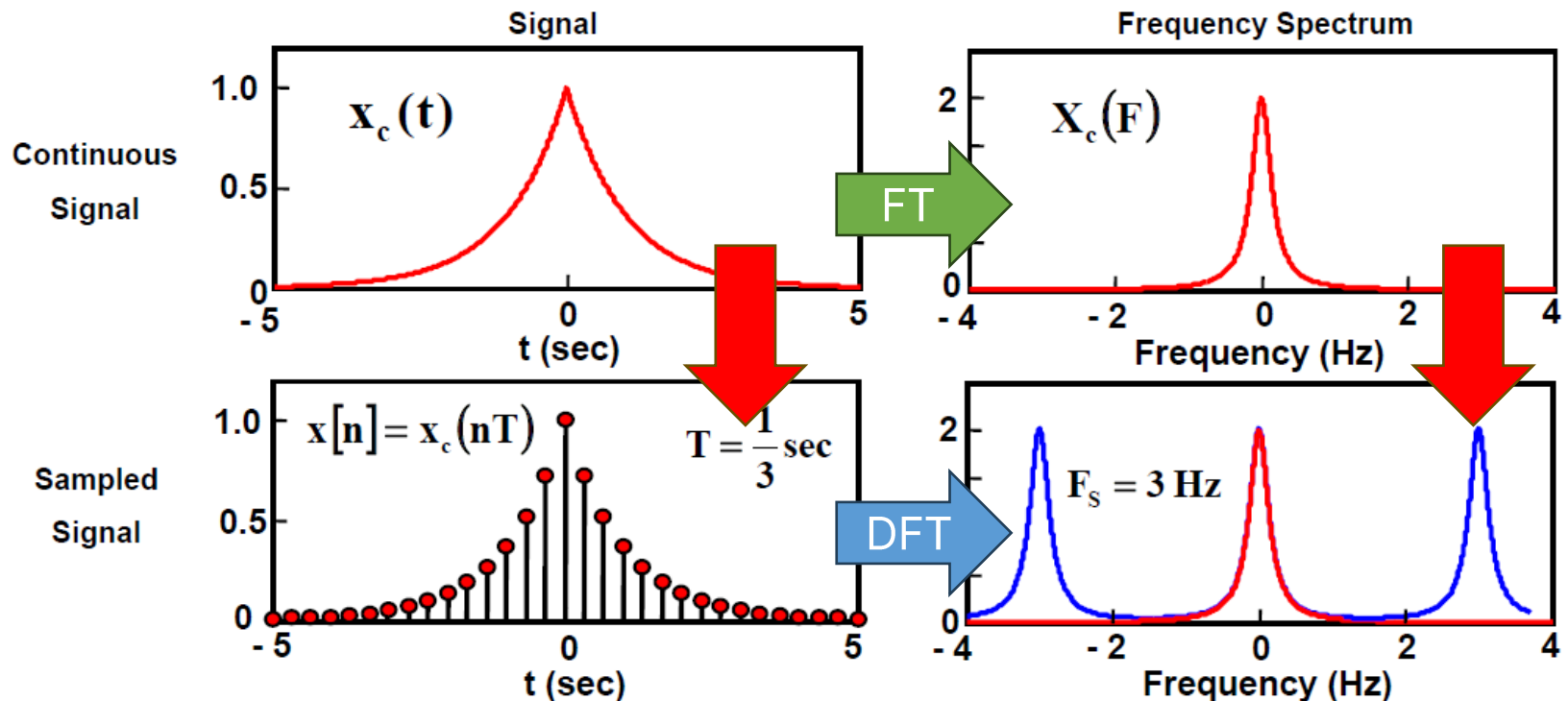
- Example of impulse response



7. Fourier Analysis of DT System

Fourier Analysis of Discrete-Time Systems

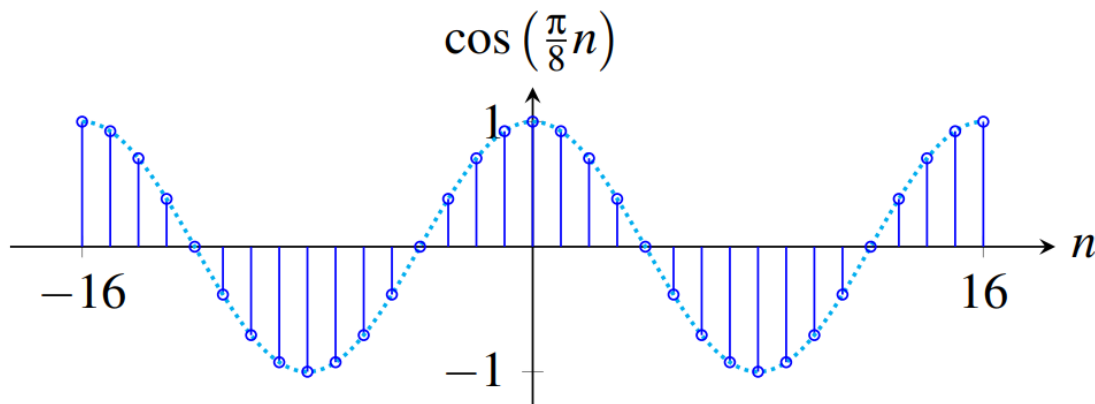
- Discrete Time Fourier Transform
 - Two systems applied in cascade way.
 - **The continuous Fourier Transform**
 - **The sampling process**



Fourier Analysis of Discrete-Time Systems

- Discrete Time Fourier Transform
 - Assume that the function is periodic with $N\Delta t$.
 - The expansion of periodicity 2π to N

$$X(\Omega_k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi k}{N}n}, \quad x(n) = \sum_{k=0}^{N-1} X(\Omega_k) e^{j\frac{2\pi n}{N}k}$$

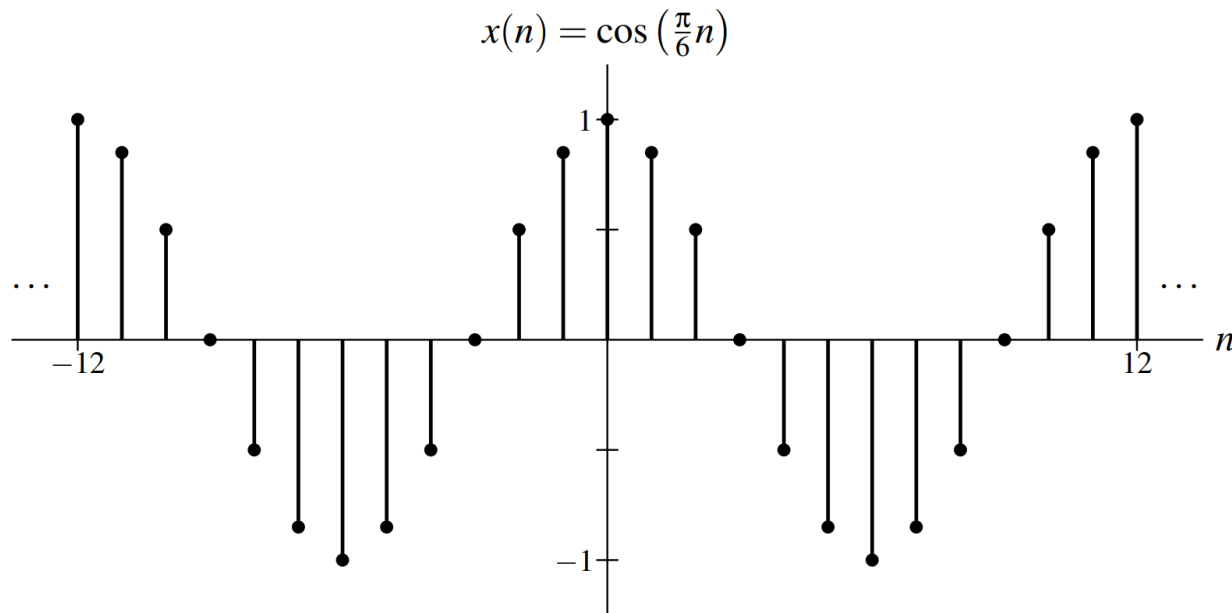


Discrete Time System

- Sinusoidal Sequences

$$x(n) = A \cos(\Omega n + \theta)$$

- A sinusoidal sequence is periodic iff. $\Omega/2\pi$ is rational.

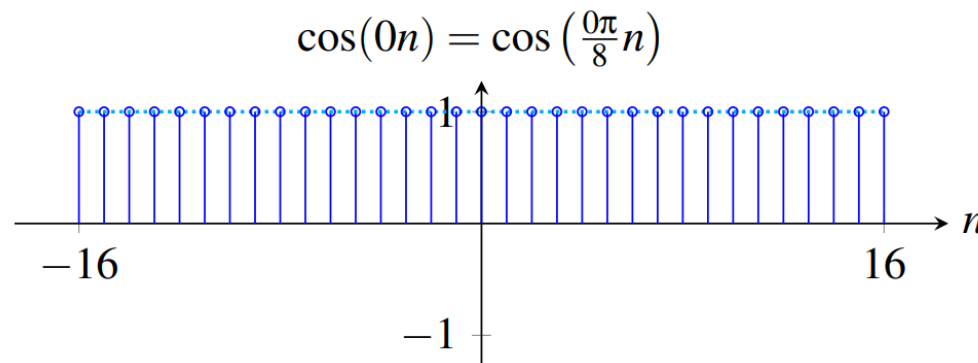


Discrete Time System

- The impact of Omega Ω

$$x(n) = A \cos(\Omega n + \theta)$$

- Consider only $0 \leq \Omega < 2\pi$
- The $x(n)$ is constant when $\Omega = 0$.

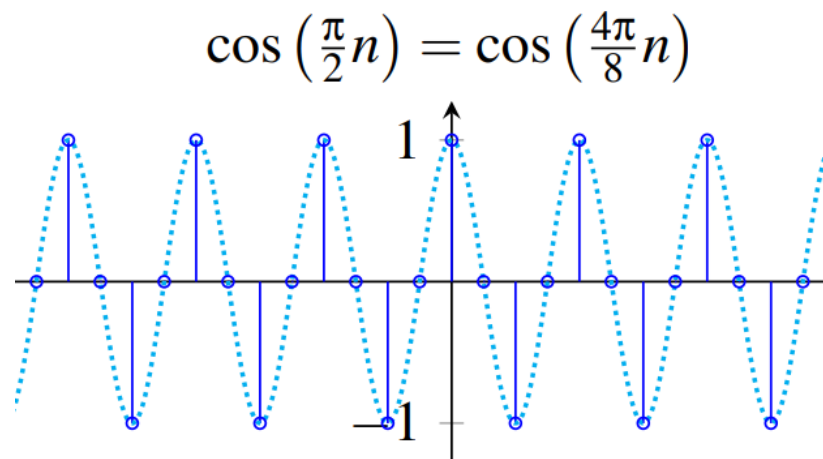
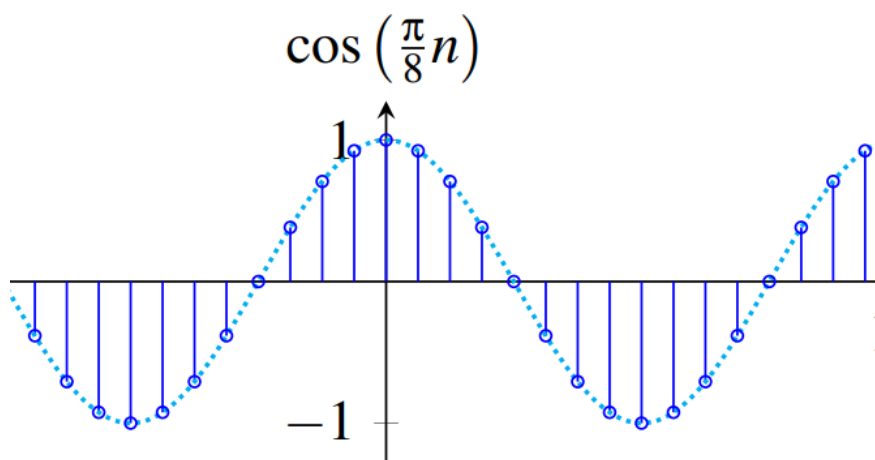


Discrete Time System

- The impact of Omega Ω

$$x(n) = A \cos(\Omega n + \theta)$$

- As Ω increases from 0 to π , the rate of oscillation of $x(n)$ increases.



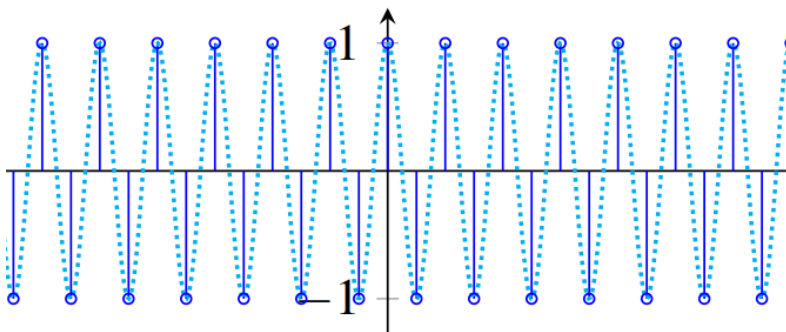
Discrete Time System

- The impact of Omega Ω

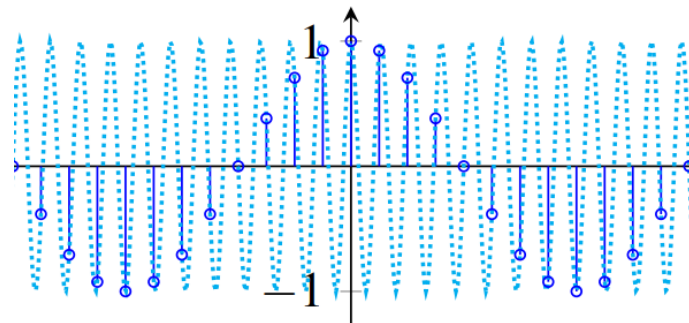
$$x(n) = A \cos(\Omega n + \theta)$$

- The **rate of oscillation** of $x(n)$ is greatest when $\Omega = \pi$.
- As Ω increases from π to 2π , the rate of oscillation of $x(n)$ **decreases**.

$$\cos(\pi n) = \cos\left(\frac{8\pi}{8}n\right)$$

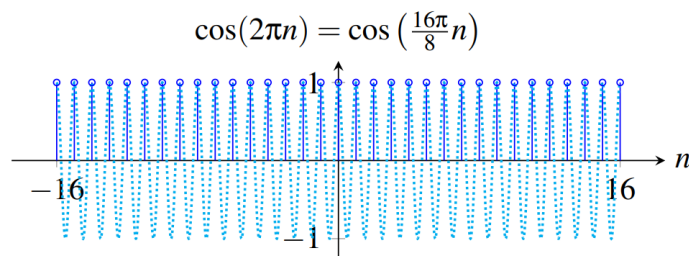
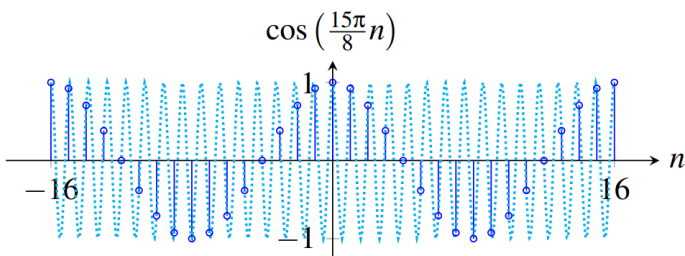
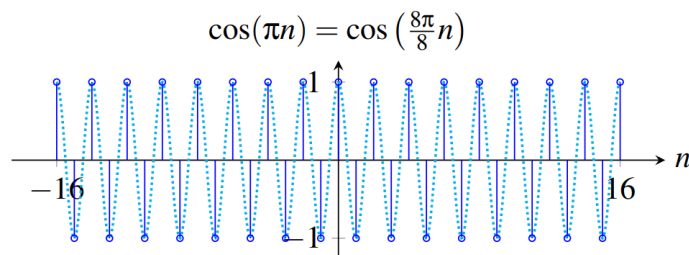
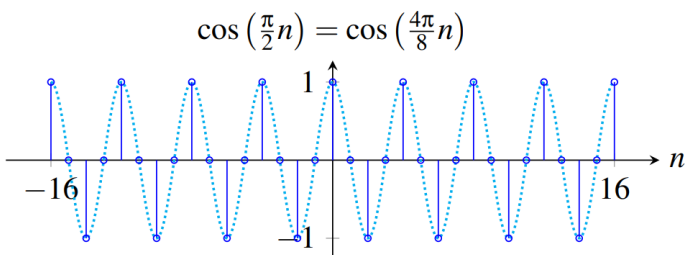
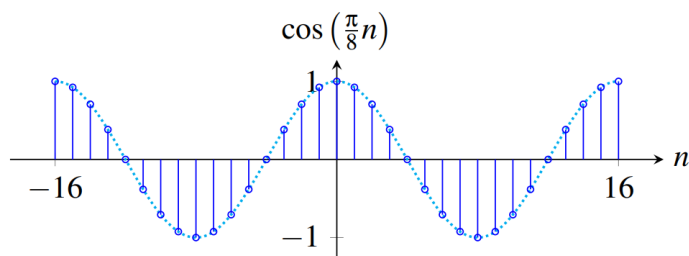
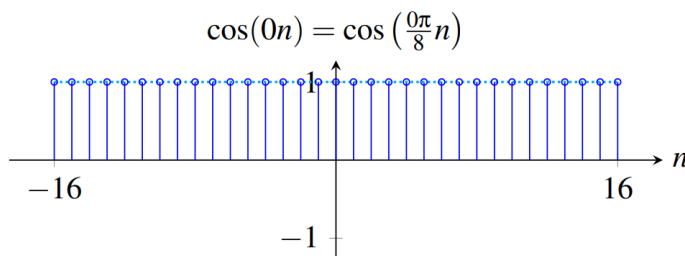


$$\cos\left(\frac{15\pi}{8}n\right)$$



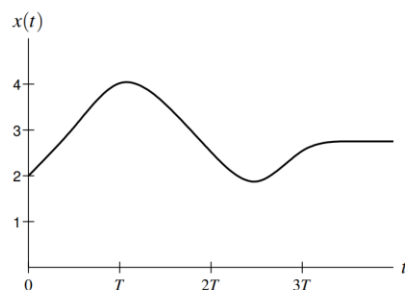
Discrete Time System

- The impact of Omega Ω

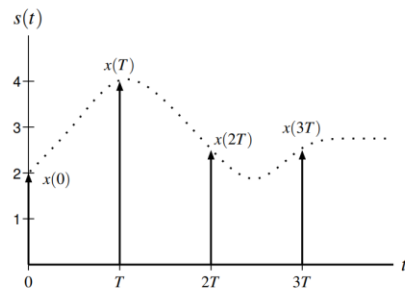
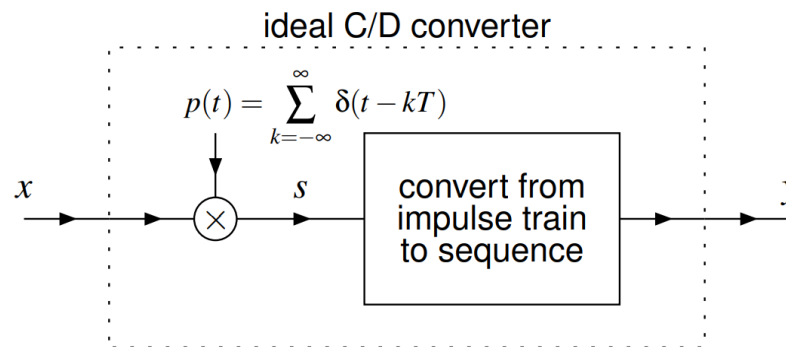


Discrete Time System

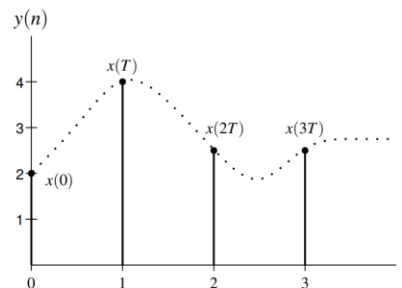
- Why is this happening?
 - (Periodic) **Sampling** : Continuous to discrete conversion.
 - The sampling period **T**



Input Function



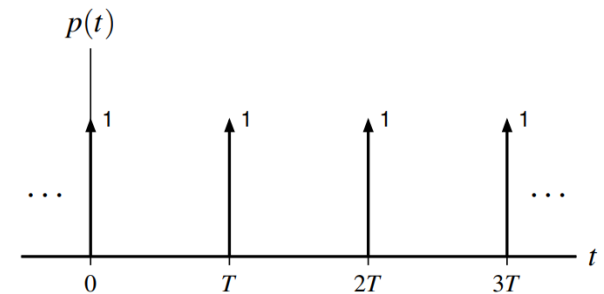
Impulse-Sampled Function
(Continuous-Time)



Output Sequence (Discrete-Time)

Discrete Time System

- The Sampling Process
 - Since the impulse train is periodic, we convert $p(t)$ by Fourier Series



Periodic Impulse Train

$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \Rightarrow \sum_{n=-\infty}^{\infty} c_n e^{i\omega_0 n t}$$

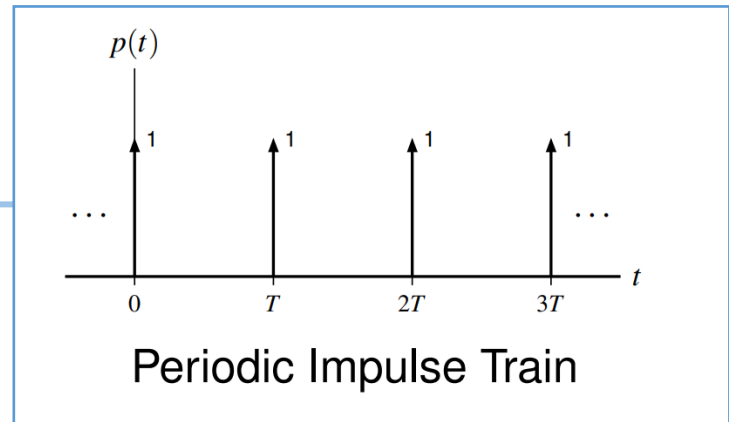
$$c_n = \frac{1}{T} \int_{-T/2}^T \delta(t - kT) e^{-i\frac{2\pi}{T} n t} dt = \frac{1}{T}$$

- Therefore,

$$p(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{i\omega_0 n t}$$

Discrete Time System

- The Sampling Process
 - Applying Fourier Transform to $p(t)$



$$p(t) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{i\omega_0 n t} \Rightarrow P(\omega) = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$$

- Note that the $F\{e^{j\omega_0 t}\} = 2\pi\delta(\omega - \omega_0)$, $\omega_0 = 2\pi/T$
- Finally, we get following.

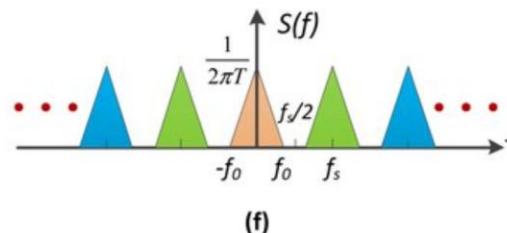
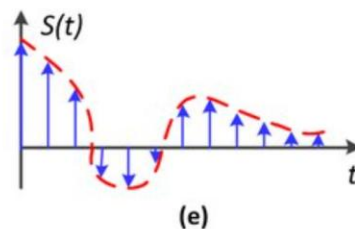
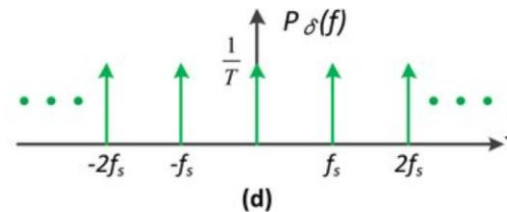
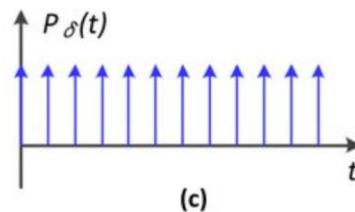
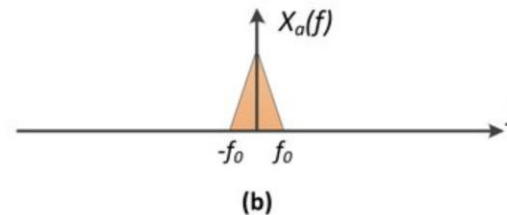
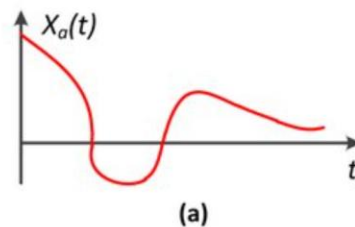
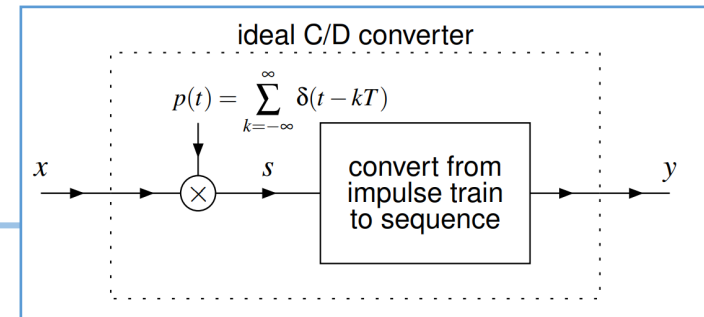
$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \Rightarrow P(\omega) = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$$

Discrete Time System

- The Sampling Process

- Sampling will perform a convolution in frequency domain.

$$x(t)h(t) \leftrightarrow \frac{1}{2\pi} X(\omega) * H(\omega)$$



Fourier Analysis of Discrete-Time Systems

- Discrete Time Fourier Transform

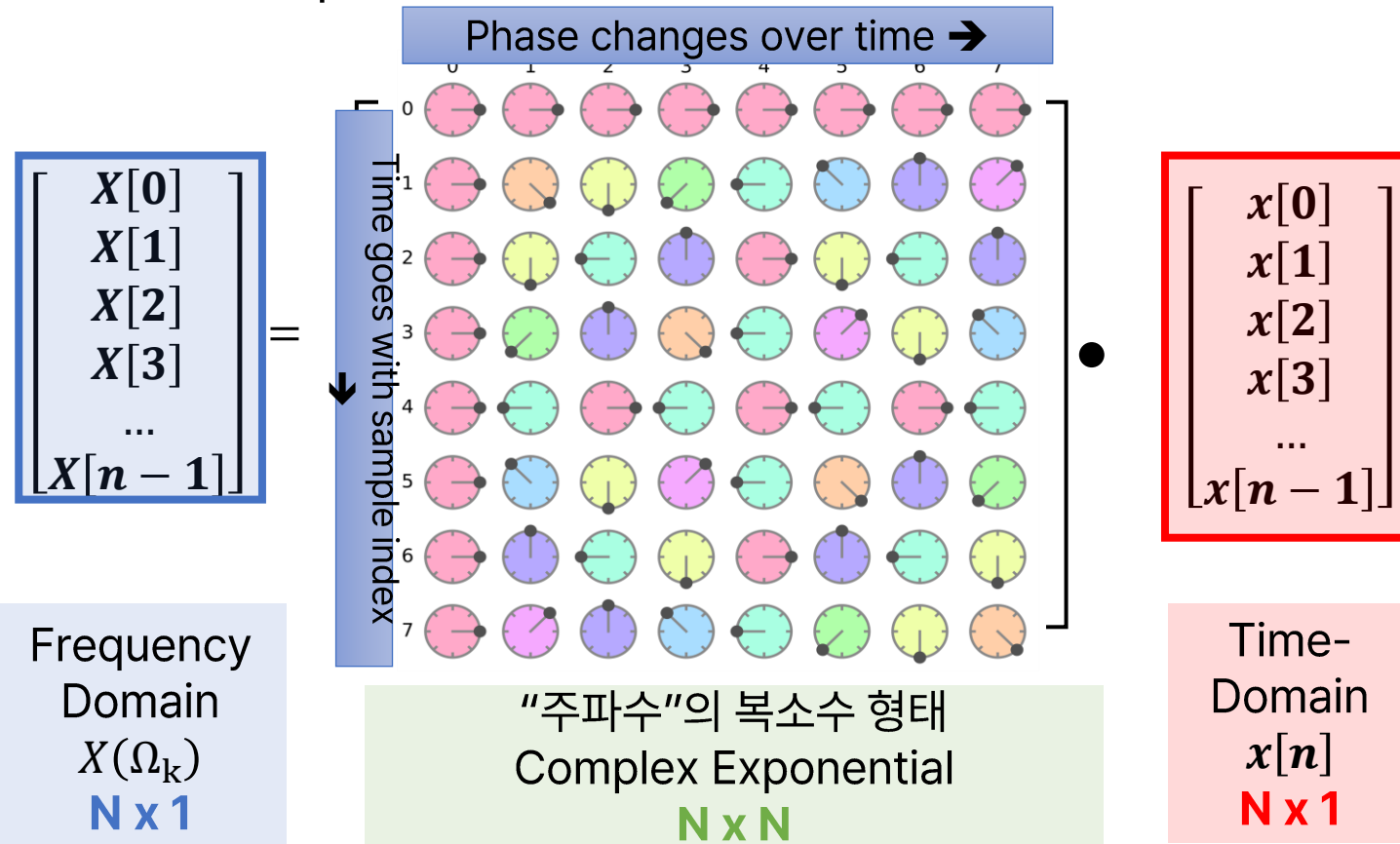
$$X(\Omega_k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi k}{N}n}$$

- We have two indexes, **n** and **k**
 - N : discrete time index
 - K : discrete frequency index
 - Note that N is the number of samples of x(n)
- The transform can be performed by multiplication of a matrix!

Fourier Analysis of Discrete-Time Systems

- Discrete Fourier Transform
 - A visual explanation

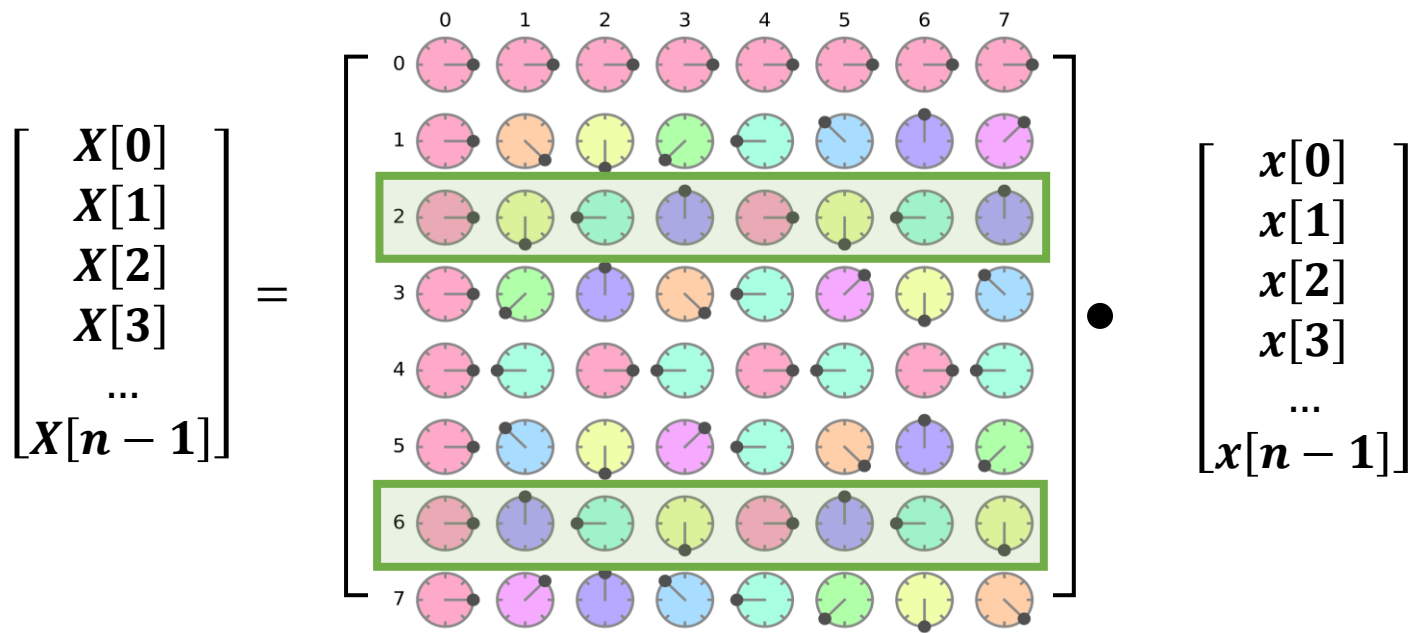
$$X(\Omega_k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi k}{N}n}$$



Fourier Analysis of Discrete-Time Systems

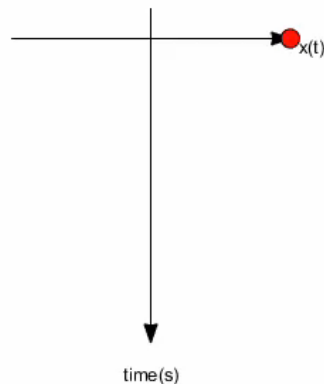
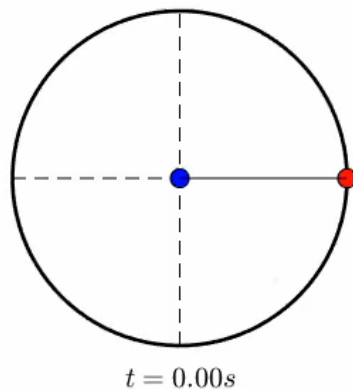
- Discrete Fourier Transform
 - The negative frequency
 - The symmetry of DFT of real signal

$$X(\Omega_k) = \sum_{n=0}^{N-1} x(n) e^{-j \frac{2\pi k}{N} n}$$



Fourier Analysis of Discrete-Time Systems

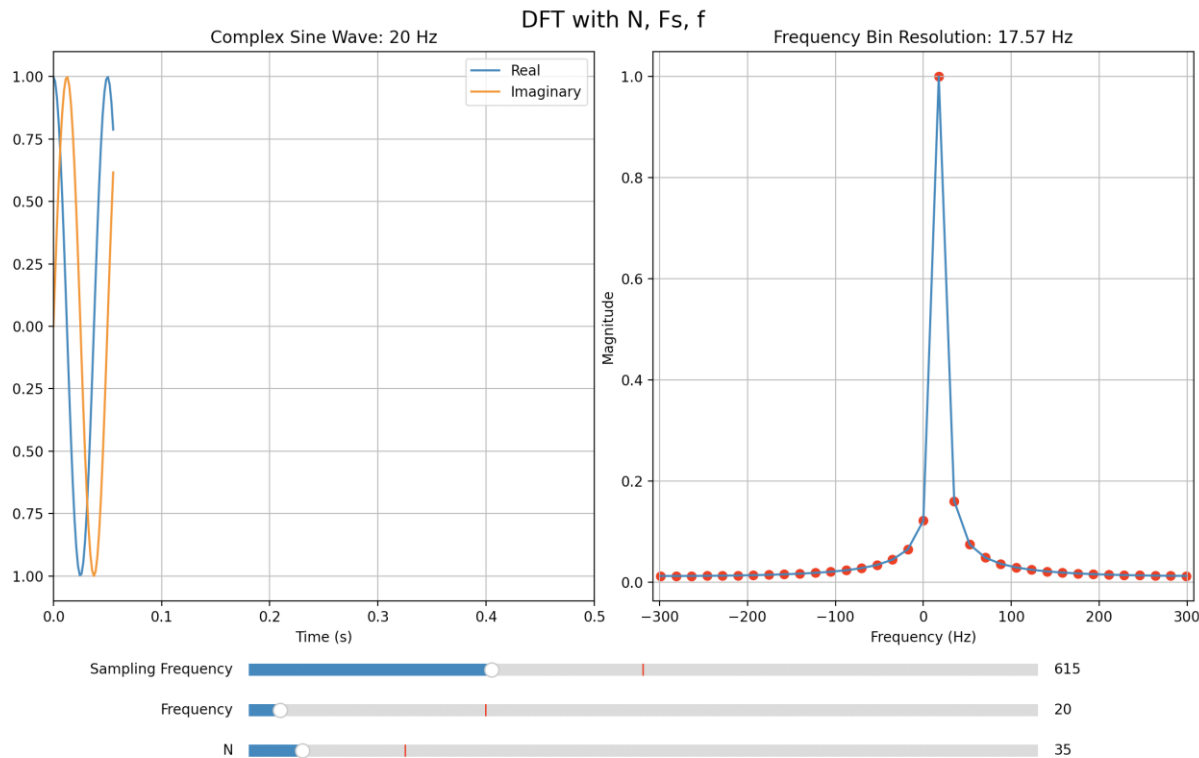
- Discrete Fourier Transform
 - Why DFT result of complex signal is not symmetrical?



$$e^{j\omega t} = \cos \omega t + j \sin \omega t$$

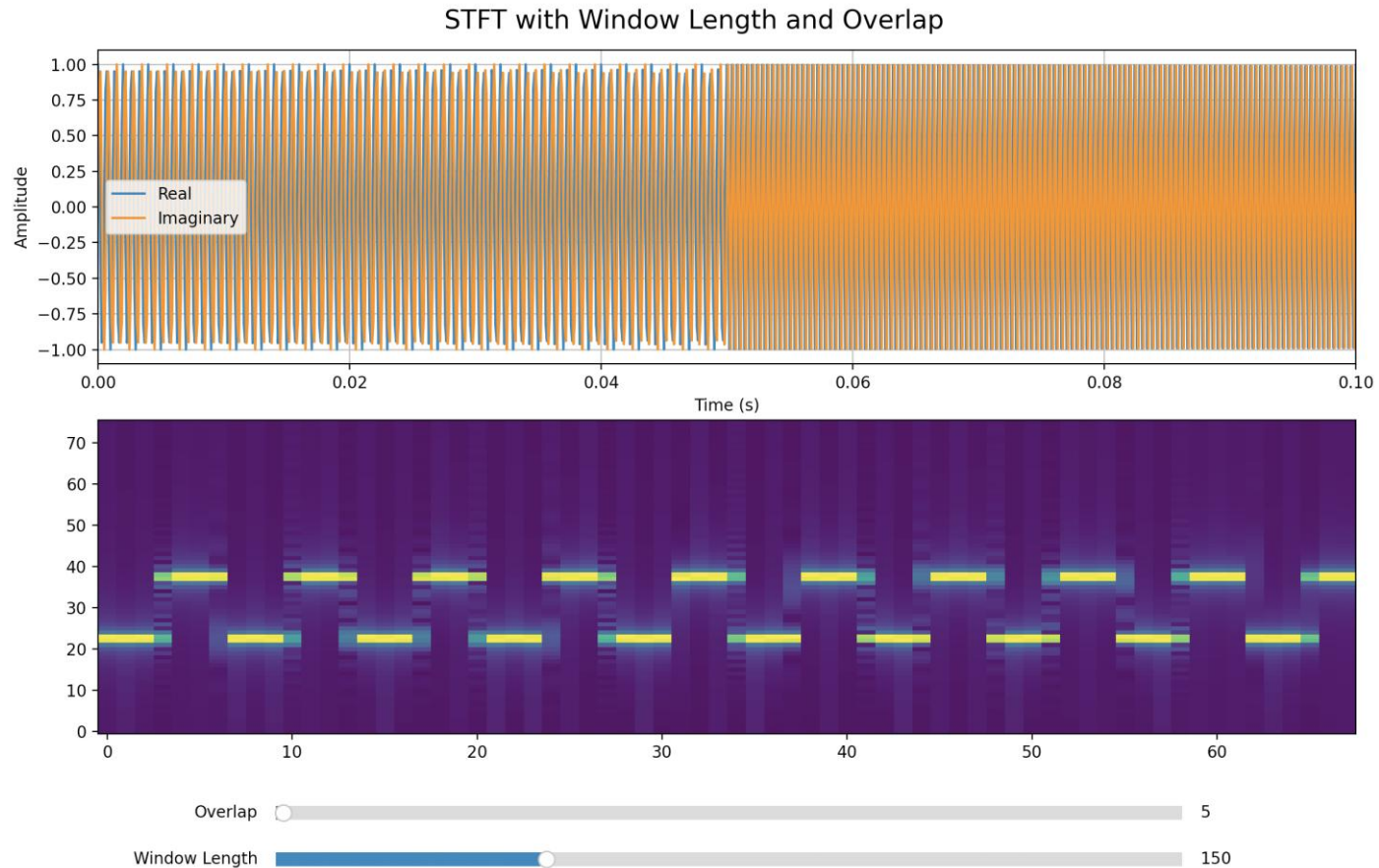
Fourier Analysis of Discrete-Time Systems

- The uncertainty of the time and frequency
 - For a fixed f_s , larger N will give small Δf (high freq. resolution)
 - For a fixed t , higher f_s will give higher f_{max}



Fourier Analysis of Discrete-Time Systems

- The time-frequency analysis



Fourier Analysis of Discrete-Time Systems

- Properties of DFT

Property	Time Domain	Frequency Domain
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(\Omega) + a_2X_2(\Omega)$
Translation	$x(n - n_0)$	$e^{-j\Omega n_0}X(\Omega)$
Modulation	$e^{j\Omega_0 n}x(n)$	$X(\Omega - \Omega_0)$
Conjugation	$x^*(n)$	$X^*(-\Omega)$
Time Reversal	$x(-n)$	$X(-\Omega)$
Upsampling	$(\uparrow M)x(n)$	$X(M\Omega)$
Downsampling	$(\downarrow M)x(n)$	$\frac{1}{M} \sum_{k=0}^{M-1} X\left(\frac{\Omega - 2\pi k}{M}\right)$
Convolution	$x_1 * x_2(n)$	$X_1(\Omega)X_2(\Omega)$
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi} \int_{2\pi} X_1(\theta)X_2(\Omega - \theta)d\theta$
Freq.-Domain Diff.	$nx(n)$	$j \frac{d}{d\Omega} X(\Omega)$
Differencing	$x(n) - x(n - 1)$	$(1 - e^{-j\Omega})X(\Omega)$
Accumulation	$\sum_{k=-\infty}^n x(k)$	$\frac{e^{j\Omega}}{e^{j\Omega} - 1} X(\Omega) + \pi X(0) \sum_{k=-\infty}^{\infty} \delta(\Omega - 2\pi k)$

Fourier Analysis of Discrete-Time Systems

- Properties of DFT

Property	
Periodicity	$X(\Omega) = X(\Omega + 2\pi)$
Parseval's Relation	$\sum_{n=-\infty}^{\infty} x(n) ^2 = \frac{1}{2\pi} \int_{2\pi} X(\Omega) ^2 d\Omega$
Even Symmetry	$x \text{ is even} \Leftrightarrow X \text{ is even}$
Odd Symmetry	$x \text{ is odd} \Leftrightarrow X \text{ is odd}$
Real / Conjugate Symmetry	$x \text{ is real} \Leftrightarrow X \text{ is conjugate symmetric}$

Thank you

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