

Signals and Systems

Department of Radio & Information Communications Engineering
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Course Overview

- Tentative Schedule

#	Topic	Assignment
1	Introduction	
2	Representing Signals	
3	Continuous-Time System	
4	Complex Numbers	
5	Fourier Series	
6	Fourier Transform	#3
7	Fourier Analysis of Discrete-Time System	

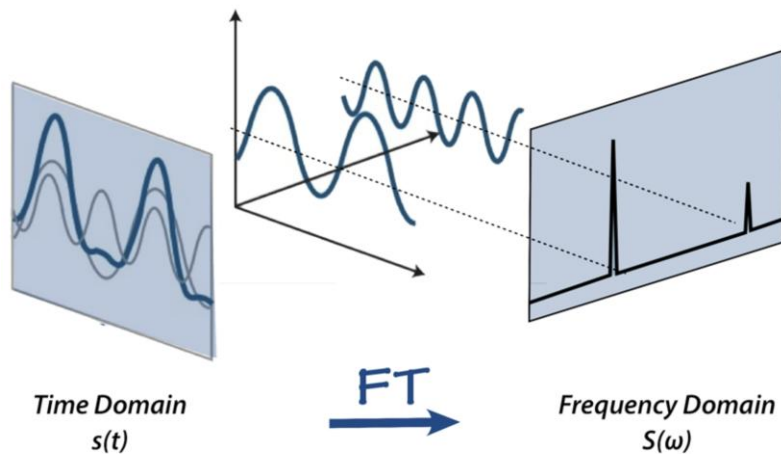
5. Fourier Series

It's not about equations.

The concept matters.

Fourier Series

- Why do we need this?
 - Image Compression
 - Wireless Communication
 - Electrical Engineering
 - Electromagnetics
 - Differential Equation
 - Quantum Physics
 - Signal Processing



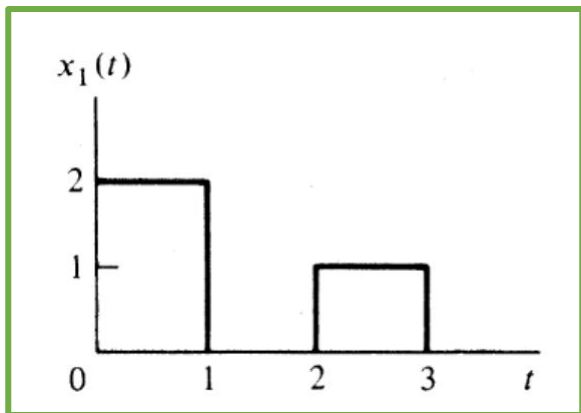
Fourier Series

- Long story short
 - Looks like we have four different but similar things to study.

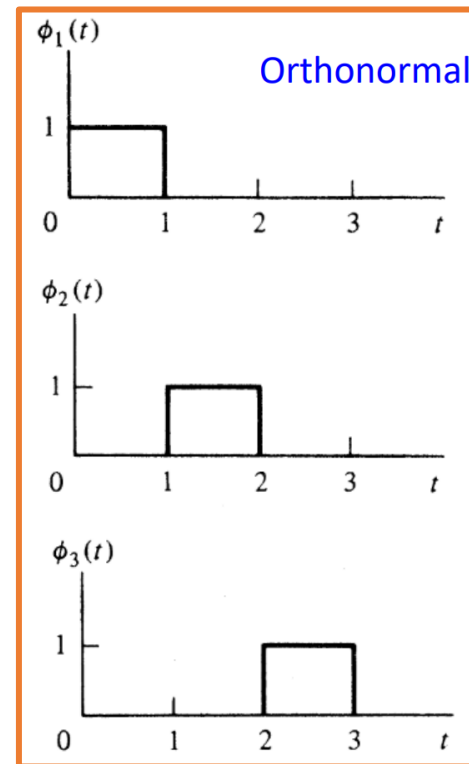
Time Duration		
Finite	Infinite	
Discrete FT (DFT)	Discrete Time FT (DTFT)	discr.
$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\omega_k n}$ $k = 0, 1, \dots, N-1$	$X(\omega) = \sum_{n=-\infty}^{+\infty} x(n)e^{-j\omega n}$ $\omega \in (-\pi, +\pi)$	time
Fourier Series (FS)	Fourier Transform (FT)	cont.
$X(k) = \int_0^P x(t)e^{-j\omega_k t} dt$ $k = -\infty, \dots, +\infty$	$X(\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t} dt$ $\omega \in (-\infty, +\infty)$	time
discrete freq. k	continuous freq. ω	t

Fourier Series

- The key idea : **Orthogonality**
 - A kind of **Linear Transformation (Multiplication of Matrix)**
 - **Original Function** = (some constant) * (Orthogonal Functions)



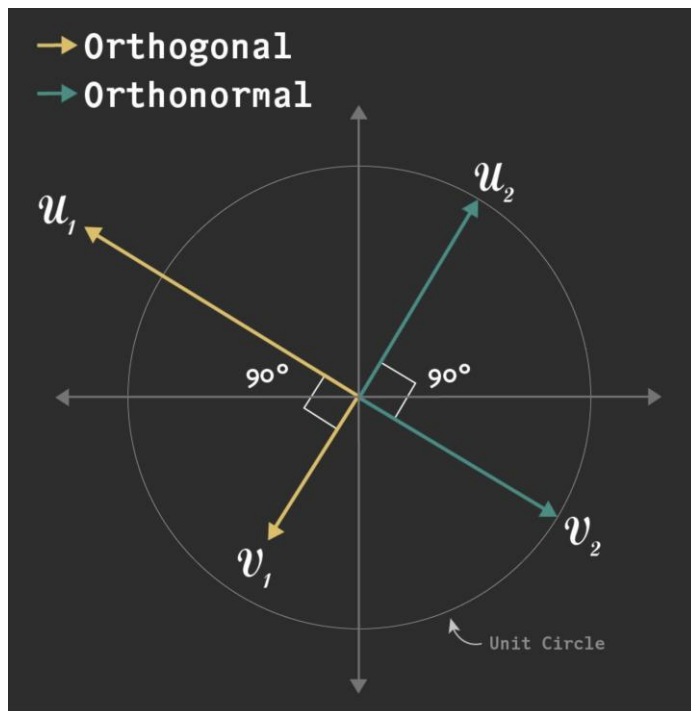
$$\begin{bmatrix} 2 & 0 & 1 \end{bmatrix}$$



Fourier Series

- Orthogonality

$$\int_0^L \sin\left(\frac{\pi nx}{L}\right) \sin\left(\frac{\pi mx}{L}\right) dx = \begin{cases} 0 & n \neq m \\ \frac{L}{2} & n = m \end{cases}, \quad n, m = 1, 2, 3, \dots$$



A Square matrix 'A' is orthogonal if

$$A^T = A^{-1}$$

(OR)

$$AA^T = A^T A = I, \text{ where}$$

- A^T = Transpose of A
- A^{-1} = Inverse of A
- I = Identity matrix of same order as 'A'

Fourier Series

- Start with Real Fourier Series
 - Target Function is periodic $\rightarrow$ Sum of periodic function

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

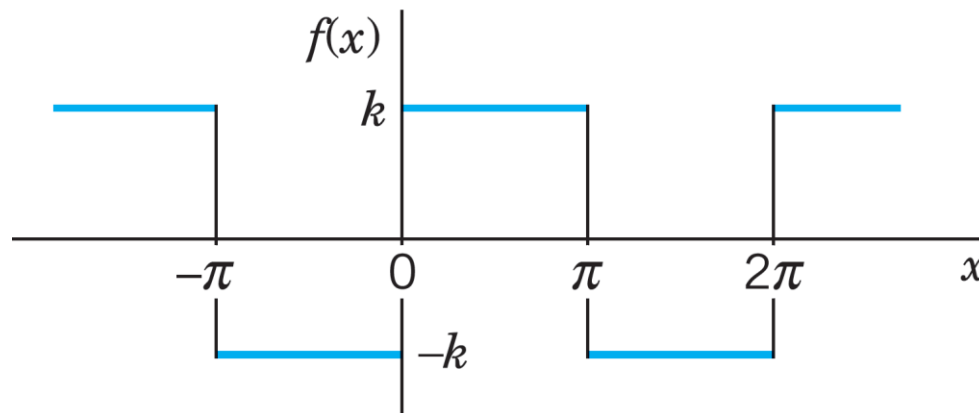
$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

Fourier Series

- Example: Periodic Rectangular Wave
 - Find Fourier coefficients of the periodic function $f(x)$

$$f(x) = \begin{cases} -k & \text{if } -\pi < x < 0 \\ k & \text{if } 0 < x < \pi \end{cases} \quad \text{and} \quad f(x + 2\pi) = f(x).$$



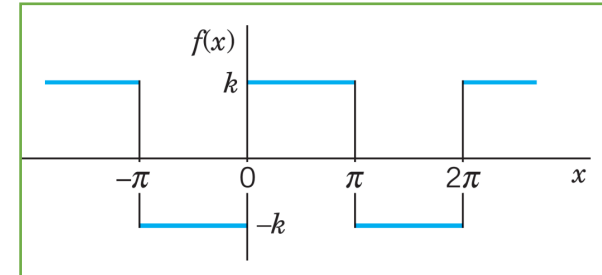
Fourier Series

- Example: Periodic Rectangular Wave
 - Find Fourier coefficients of the periodic function $f(x)$

$$\begin{aligned} a_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{1}{\pi} \left[\int_{-\pi}^0 (-k) \cos nx \, dx + \int_0^{\pi} k \cos nx \, dx \right] \\ &= \frac{1}{\pi} \left[-k \frac{\sin nx}{n} \Big|_{-\pi}^0 + k \frac{\sin nx}{n} \Big|_0^{\pi} \right] = 0 \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx = \frac{1}{\pi} \left[\int_{-\pi}^0 (-k) \sin nx \, dx + \int_0^{\pi} k \sin nx \, dx \right] \\ &= \frac{1}{\pi} \left[k \frac{\cos nx}{n} \Big|_{-\pi}^0 - k \frac{\cos nx}{n} \Big|_0^{\pi} \right]. \end{aligned}$$

$$= \frac{k}{n\pi} [\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0] = \frac{2k}{n\pi} (1 - \cos n\pi).$$



Fourier Series

- Example: Periodic Rectangular Wave

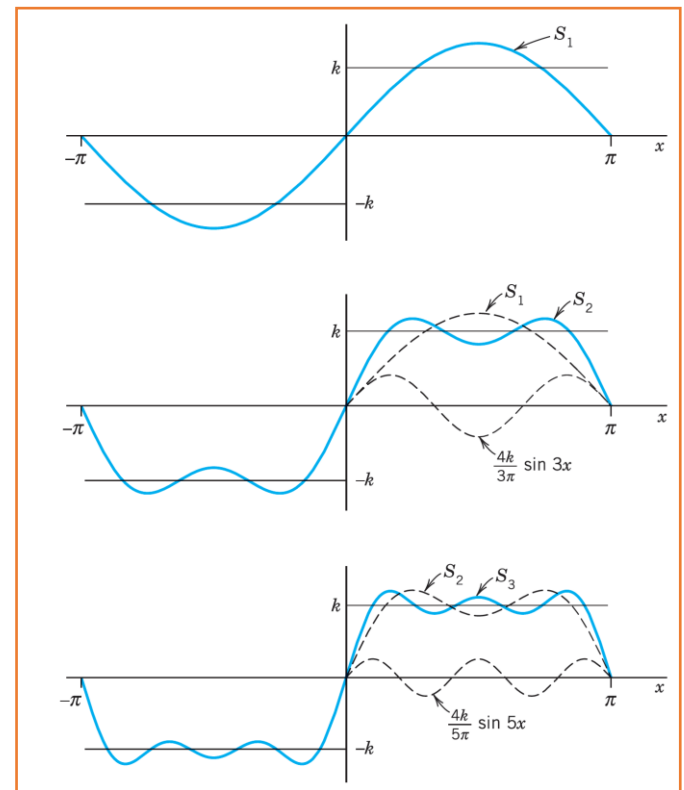
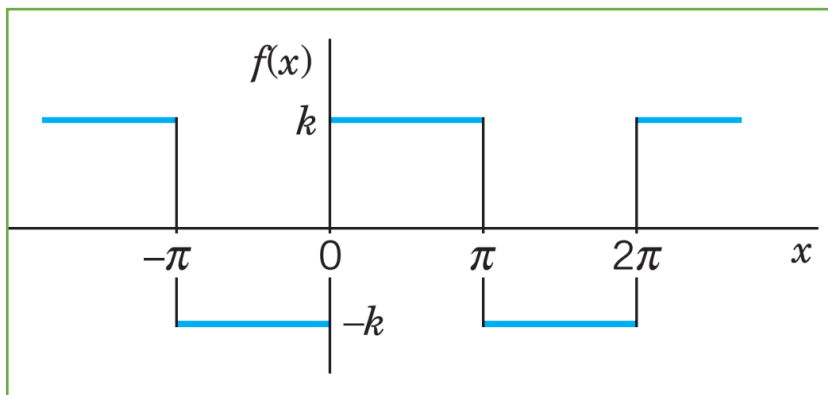
- $f(x) = \frac{4k}{\pi} \left(\sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \dots \right)$

- Let the partial sum

- $S_1 = \frac{4k}{\pi} \sin x$

- $S_2 = \frac{4k}{\pi} \left(\sin x + \frac{1}{3} \sin 3x \right)$

- ...



Fourier Series

- Derivation of the Euler Formulas

Orthogonality of the Trigonometric System (3)

The trigonometric system (3) is orthogonal on the interval $-\pi \leq x \leq \pi$ (hence also on $0 \leq x \leq 2\pi$ or any other interval of length 2π because of periodicity); that is, the integral of the product of any two functions in (3) over that interval is 0, so that for any integers n and m ,

$$(a) \quad \int_{-\pi}^{\pi} \cos nx \cos mx \, dx = 0 \quad (n \neq m)$$

$$(9) \quad (b) \quad \int_{-\pi}^{\pi} \sin nx \sin mx \, dx = 0 \quad (n \neq m)$$

$$(c) \quad \int_{-\pi}^{\pi} \sin nx \cos mx \, dx = 0 \quad (n \neq m \text{ or } n = m).$$

Fourier Series

- Before we prove Theorem 1.
 - For integer $m \neq n$,

$$\int_{-\pi}^{\pi} \cos nx \cos mx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos (n + m)x \, dx + \frac{1}{2} \int_{-\pi}^{\pi} \cos (n - m)x \, dx$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos (n - m)x \, dx - \frac{1}{2} \int_{-\pi}^{\pi} \cos (n + m)x \, dx.$$

- For any integer m and n , (Why?)

$$\int_{-\pi}^{\pi} \sin nx \cos mx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \sin (n + m)x \, dx + \frac{1}{2} \int_{-\pi}^{\pi} \sin (n - m)x \, dx = 0 + 0.$$

$$\begin{aligned}\cos A \cos B &= \frac{1}{2} [\cos(A + B) + \cos(A - B)] \\ \sin A \sin B &= -\frac{1}{2} [\cos(A + B) - \cos(A - B)] \\ \cos A \sin B &= \frac{1}{2} [\sin(A + B) - \sin(A - B)] \\ \sin A \cos B &= \frac{1}{2} [\sin(A + B) + \sin(A - B)]\end{aligned}$$

Fourier Series

- Proof of Theorem 1.
 - Apply integration on below equation.

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$\int_{-\pi}^{\pi} f(x) dx = a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx dx + b_n \int_{-\pi}^{\pi} \sin nx dx \right).$$

1

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx$$

2

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

3

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$$

Fourier Series

- Proof of Theorem 1.
 - Apply integration on below equation.

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

2

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$\int_{-\pi}^{\pi} f(x) \, dx = a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx \, dx + b_n \int_{-\pi}^{\pi} \sin nx \, dx \right).$$

× cos mx

$$\int_{-\pi}^{\pi} f(x) \cos mx \, dx = \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right] \cos mx \, dx.$$

Fourier Series

- Proof of Theorem 1.
 - Apply integration on below equation.

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

3

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$\int_{-\pi}^{\pi} f(x) \, dx = a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx \, dx + b_n \int_{-\pi}^{\pi} \sin nx \, dx \right).$$

× sin mx

$$\int_{-\pi}^{\pi} f(x) \sin mx \, dx = \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \right] \sin mx \, dx.$$

Fourier Series

- Sinusoid to Complex Exponential

- From Euler's Formulas,

$$\cos \varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2}, \quad \sin \varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i},$$

- The derivation is simple.

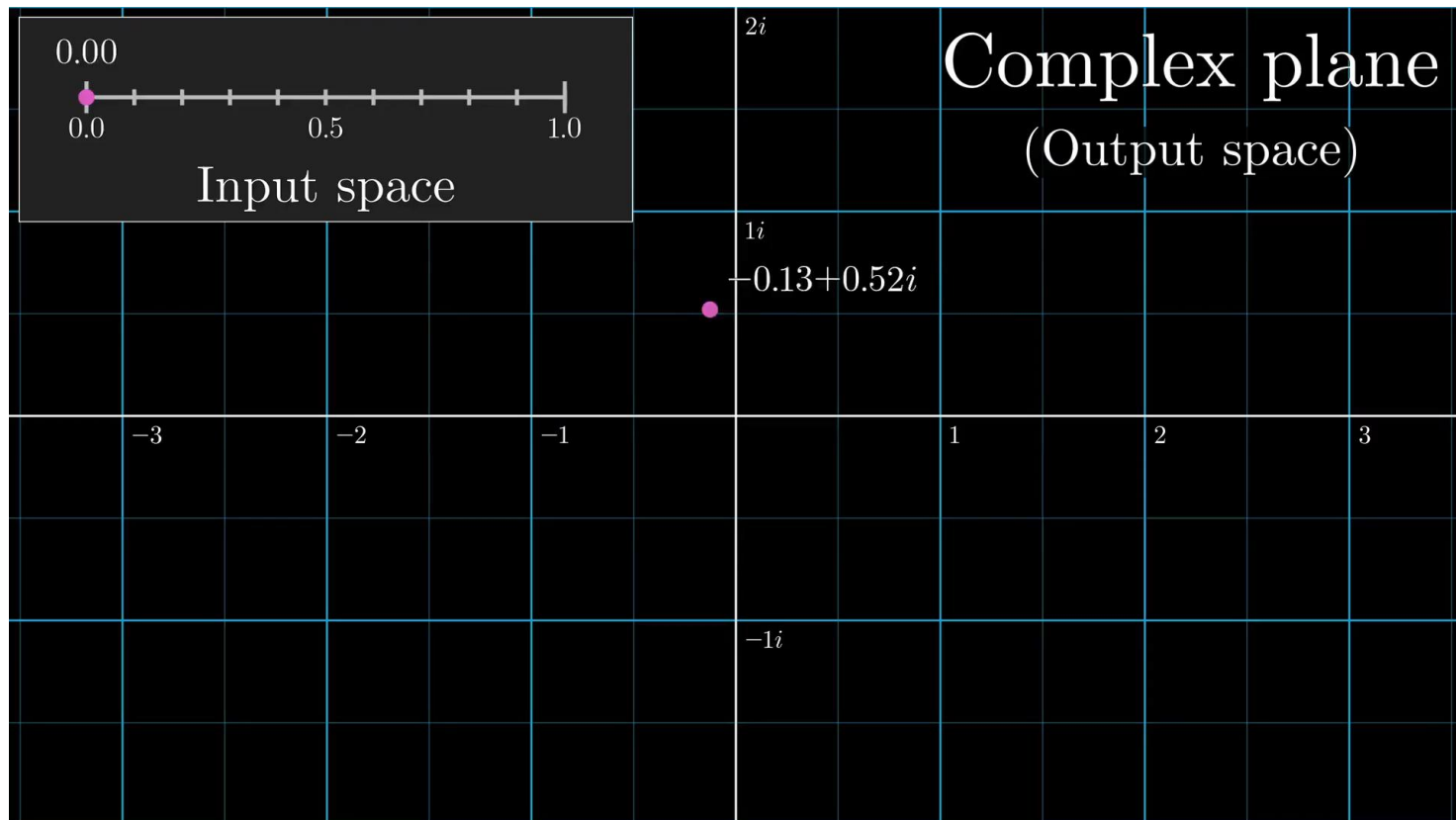
$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \frac{e^{inx} + e^{-inx}}{2} + b_n \frac{e^{inx} - e^{-inx}}{2i} \right) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{a_n - ib_n}{2} e^{inx} + \sum_{n=1}^{\infty} \frac{a_n + ib_n}{2} e^{-inx} \\ &= \sum_{n=-\infty}^{\infty} c_n e^{inx}. \end{aligned}$$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} dx, \quad n = 0, \pm 1, \pm 2, \dots$$

Fourier Series

- Visualizing the Complex Fourier Series
 - The output is a complex number, a point.

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

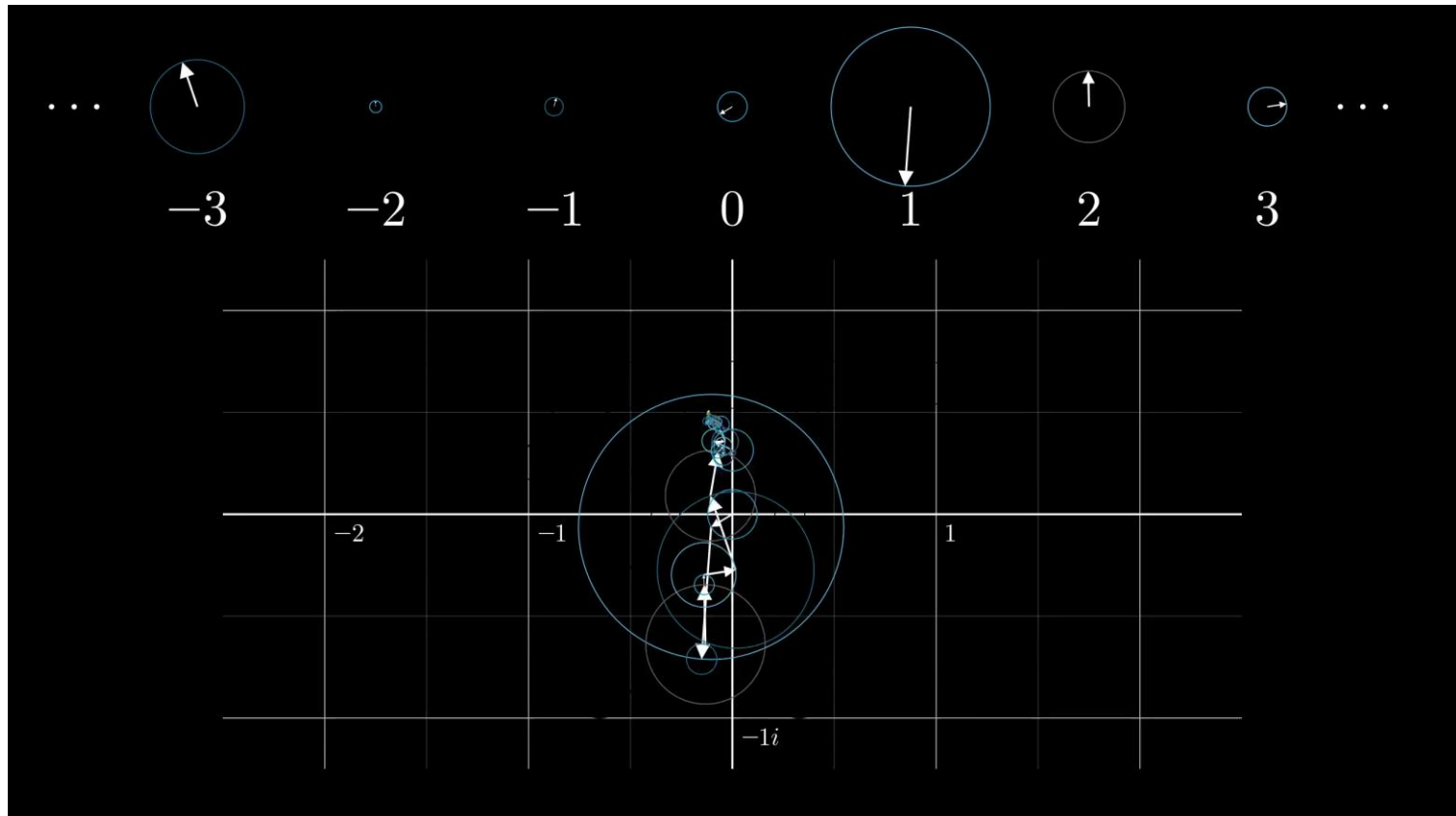


Fourier Series

- Visualizing the Complex Fourier Series

- Vectors with some radius, rotation speed.**

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

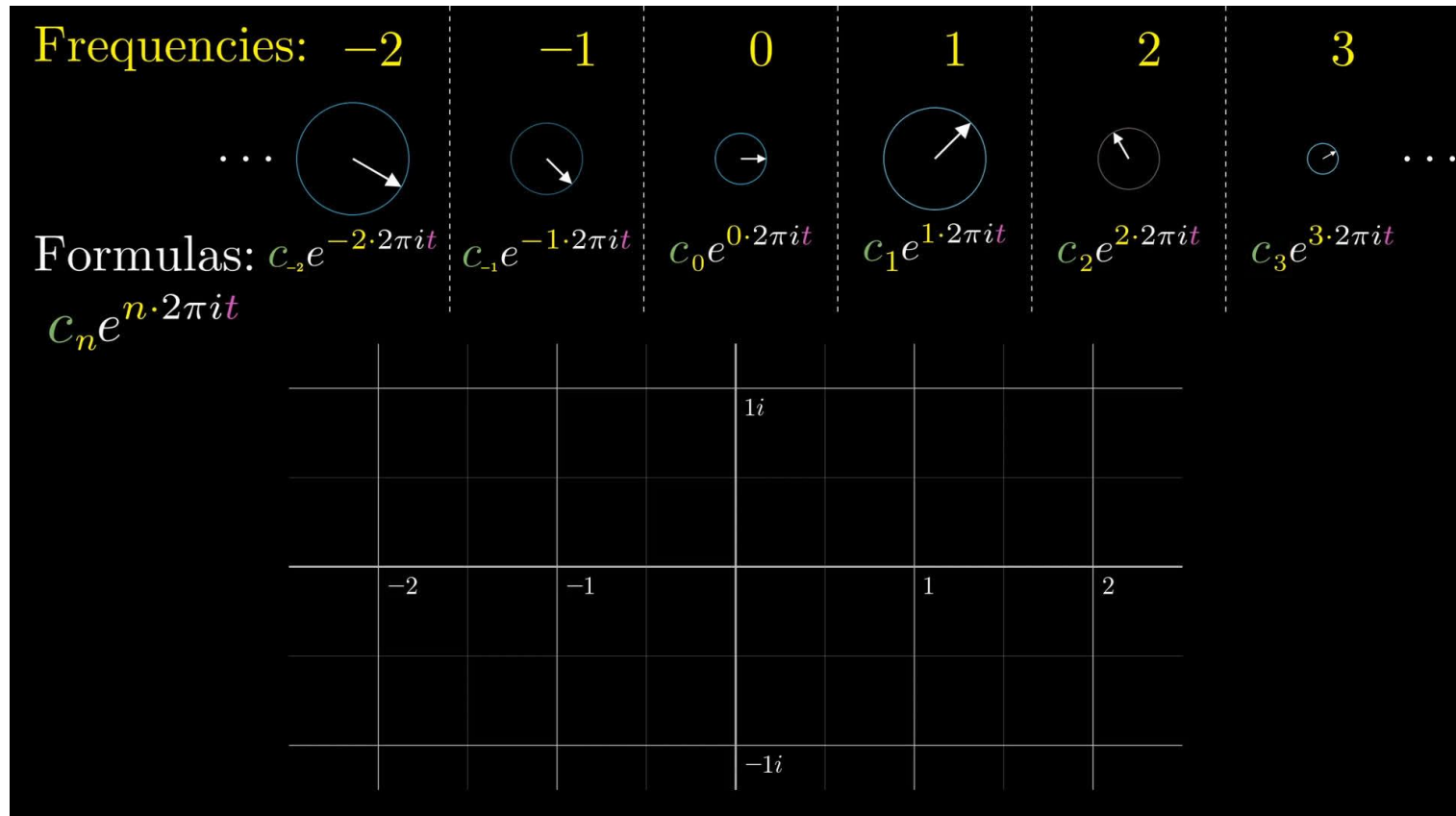


Fourier Series

- Visualizing the Complex Fourier Series

- Vectors with some **radius**, **rotation speed**.

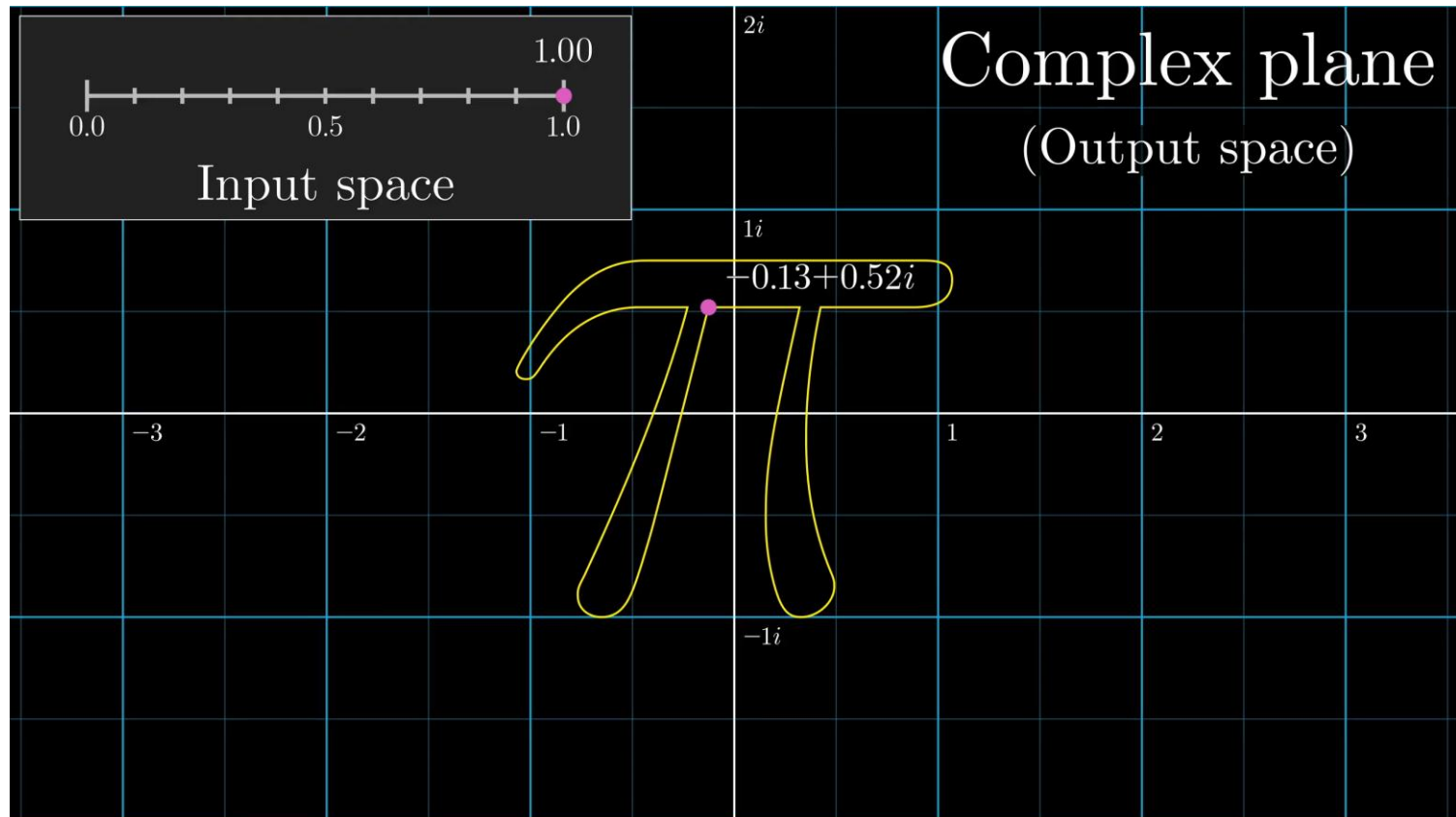
$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$



Fourier Series

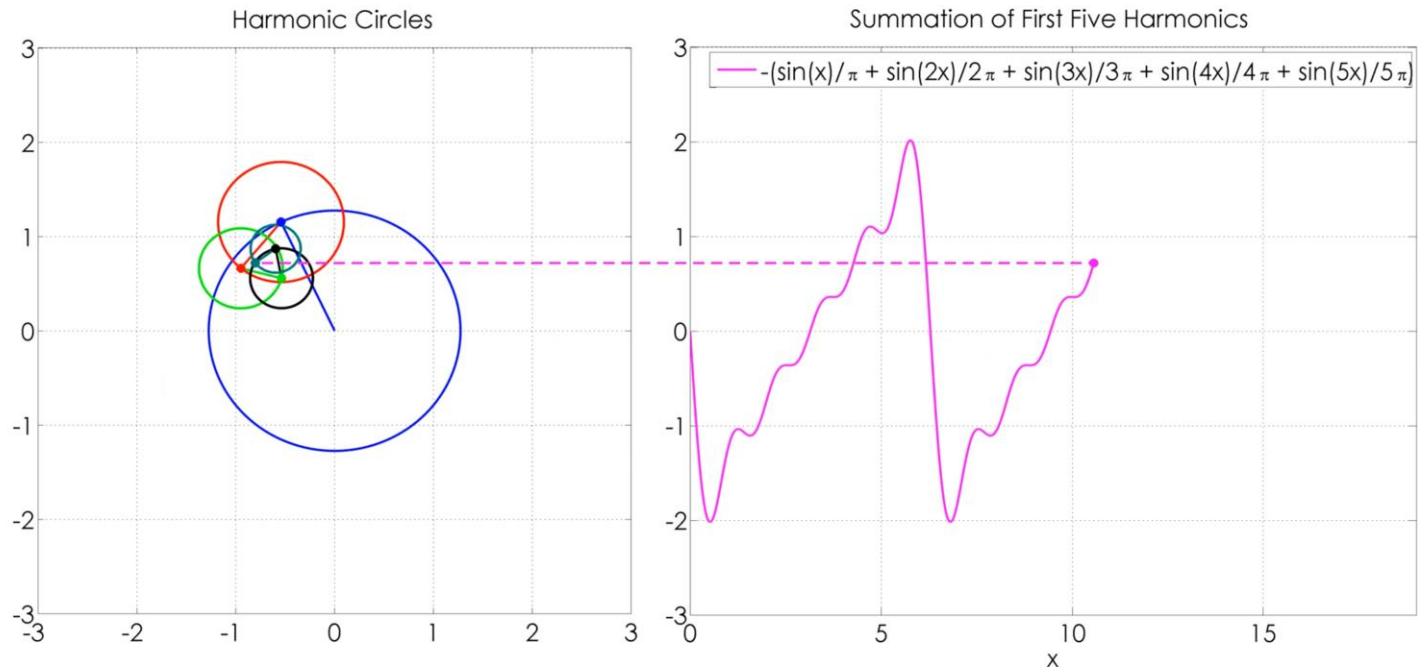
- Visualizing the Complex Fourier Series
 - The real and imaginary parts.

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$



Fourier Series

- Interactive Fourier Series of Saw-tooth Function
 - <http://www.jezzamon.com/fourier/>
 - <https://kr.mathworks.com/matlabcentral/fileexchange/49533-fourier-series-animation-using-harmonic-circles>



Fourier Series

- Continuous Fourier **Series** & **Transform**
 - **Series**: Periodic function with "discrete" exponentials

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

- **Transform**: Nonperiodic function, with continuous superposition of complex exponentials

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx.$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx} dw$$

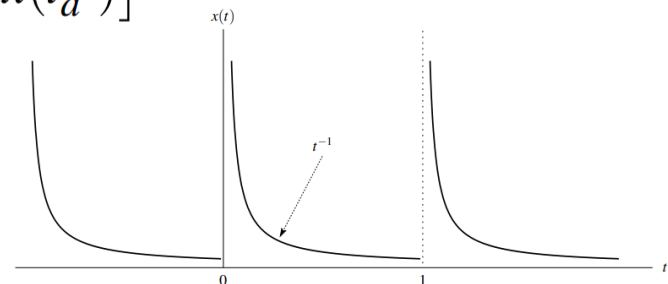
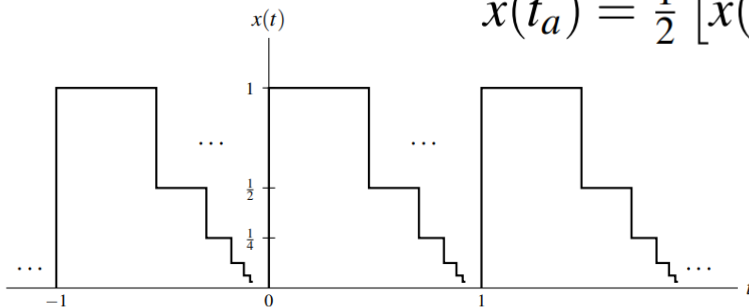
Fourier Series

- Dirichlet Conditions

- 1) over a single period, $x(t)$ is **absolutely integrable** (i.e., $\int_T |x(t)|dt < \infty$)
- 2) over a single period, $x(t)$ has a **finite number of maxima and minima** (i.e., x is of bounded variation)
- 3) over any finite interval, $x(t)$ has a **finite number of discontinuities**, each of which is finite.

- If the signal $x(t)$ satisfies above conditions,
- The Fourier series is convergent except at any point where $x(t)$ is discontinuous.
- At each point of discontinuity, it converges to

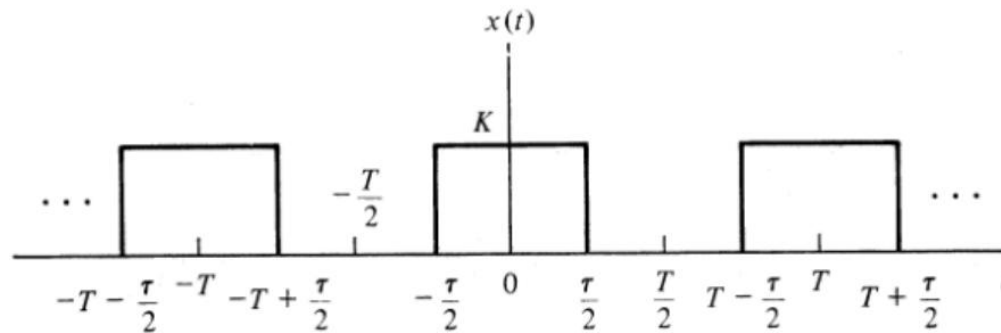
$$\tilde{x}(t_a) = \frac{1}{2} [x(t_a^-) + x(t_a^+)]$$



Fourier Series

- Dirichlet Conditions

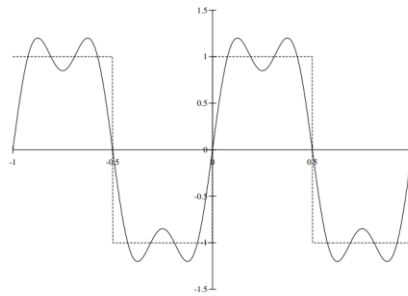
- Example, Check if the function satisfies Dirichlet conditions.



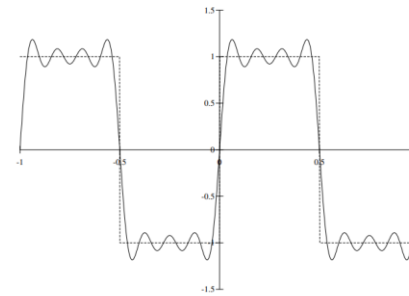
- 1) over a single period, $x(t)$ is **absolutely integrable** (i.e., $\int_T |x(t)| dt < \infty$)
- 2) over a single period, $x(t)$ has a **finite number of maxima and minima** (i.e., x is of bounded variation)
- 3) over any finite interval, $x(t)$ has a **finite number of discontinuities**, each of which is finite.

Fourier Series

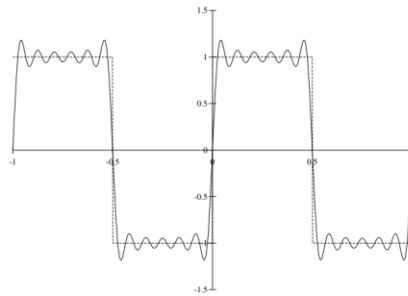
- Gibbs Phenomenon
 - If the function has discontinuities, does not converge uniformly.
 - i.e., Fourier series converge very slowly near discontinuities.



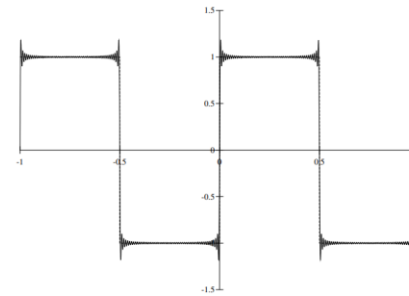
Fourier series truncated after the 3rd harmonic components



Fourier series truncated after the 7th harmonic components



Fourier series truncated after the 11th harmonic components



Fourier series truncated after the 101st harmonic components

Fourier Series

- Properties of Fourier Series

- Note that some notations are changed.

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}. \quad c_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt,$$

$$x(t) \xleftrightarrow{\text{CTFS}} a_k \quad \text{and} \quad y(t) \xleftrightarrow{\text{CTFS}} b_k$$

Property	Time Domain	Fourier Domain
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha a_k + \beta b_k$
Translation	$x(t - t_0)$	$e^{-jk(2\pi/T)t_0} a_k$
Modulation	$e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Reflection	$x(-t)$	a_{-k}
Conjugation	$x^*(t)$	a_{-k}^*
Periodic Convolution	$x \circledast y(t)$	$T a_k b_k$
Multiplication	$x(t)y(t)$	$\sum_{n=-\infty}^{\infty} a_n b_{k-n}$

$$\begin{aligned}\cos A \cos B &= \frac{1}{2} [\cos(A + B) + \cos(A - B)] \\ \sin A \sin B &= -\frac{1}{2} [\cos(A + B) - \cos(A - B)] \\ \cos A \sin B &= \frac{1}{2} [\sin(A + B) - \sin(A - B)] \\ \sin A \cos B &= \frac{1}{2} [\sin(A + B) + \sin(A - B)]\end{aligned}$$

Series

Property	Time Domain	Fourier Domain
Linearity	$\alpha x(t) + \beta y(t)$	$\alpha a_k + \beta b_k$
Translation	$x(t - t_0)$	$e^{-jk(2\pi/T)t_0} a_k$
Modulation	$e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Reflection	$x(-t)$	a_{-k}
Conjugation	$x^*(t)$	a_{-k}^*
Periodic Convolution	$x \circledast y(t)$	$T a_k b_k$
Multiplication	$x(t)y(t)$	$\sum_{n=-\infty}^{\infty} a_n b_{k-n}$

Fourier Series

- Properties of Fourier Series
 - Translation (Time Shifting)

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}. \quad c_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt,$$

$$x(t - t_0) \xrightarrow{\text{CTFS}} e^{-jk\omega_0 t_0} c_k = e^{-jk(2\pi/T)t_0} c_k,$$

- In other words, time shifting changes the argument(phase).
- The magnitude remains same.

Fourier Series

- Properties of Fourier Series
 - Modulation (Frequency Shifting)

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}. \quad c_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt,$$

$$e^{jM(2\pi/T)t} x(t) = e^{jM\omega_0 t} x(t) \xleftrightarrow{\text{CTFS}} c_{k-M},$$

- Multiplying a periodic exponential shifts the Fourier series sequence (frequency)

Fourier Series

- Properties of Fourier Series
 - Time Reversal (Reflection)

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}. \quad c_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt,$$

$$x(-t) \xleftrightarrow{\text{CTFS}} c_{-k}.$$

- Time reversal is equivalent to a coefficient index reversal (negative frequency)

Fourier Series

- Properties of Fourier Series
 - Periodic Convolution

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}. \quad c_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt,$$

$$x \circledast y(t) \xleftrightarrow{\text{CTFS}} T a_k b_k.$$

- The convolution in time domain corresponds to multiplication of the Fourier-series coefficient sequences.

Fourier Series

- Properties of Fourier Series
 - Multiplication

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}. \quad c_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt,$$

$$x(t)y(t) \xleftrightarrow{\text{CTFS}} \sum_{n=-\infty}^{\infty} a_n b_{k-n}$$

- The multiplication of two periodic functions corresponds to the DT convolution of Fourier Series coefficient sequences.

Fourier Series

- Properties of Fourier Series
 - Note that some notations are changed.

Property	
Parseval's Relation	$\frac{1}{T} \int_T x(t) ^2 dt = \sum_{k=-\infty}^{\infty} a_k ^2$
Even Symmetry	x is even $\Leftrightarrow a$ is even
Odd Symmetry	x is odd $\Leftrightarrow a$ is odd
Real / Conjugate Symmetry	x is real $\Leftrightarrow a$ is conjugate symmetric

Fourier Series

- Properties of Fourier Series
 - Parseval's Relation

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}.$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_0 t}. \quad c_k = \frac{1}{T} \int_T x(t) e^{-jk\omega_0 t} dt,$$

$$\frac{1}{T} \int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2.$$

- The amount of energy is preserved!

Thank you

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