

Signals and Systems

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Course Overview

- Tentative Schedule

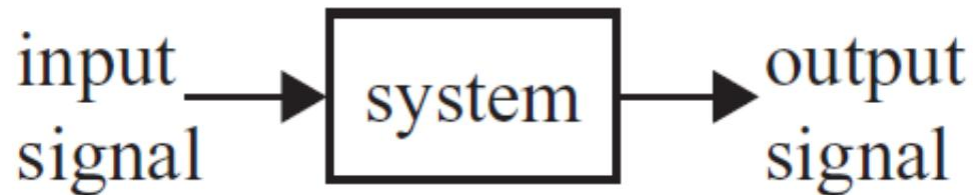
#	Topic	Assignment
1	Introduction	
2	Representing Signals	
3	Continuous-Time System	
4	Complex Numbers	#2
5	Fourier Series	
6	Fourier Transform	
7	Fourier Analysis of Discrete-Time System	

3. Continuous-Time System

Systems, Functions, ...

Continuous-Time System

- What are Systems?
 - An entity that processes and produces a set of signals.



- Linear / Nonlinear
- Causal / Non-causal
- Time-varying / Time-invariant
- Invertible / Non-invertible
- Memory / Memoryless
- Stable / Unstable

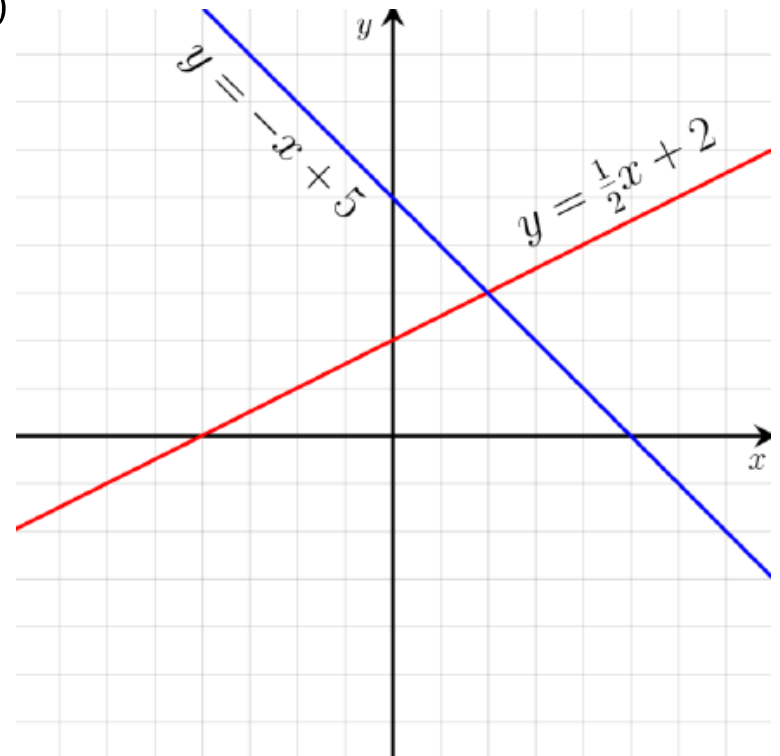
Continuous-Time System

- Linear and Nonlinear System

- Linear function (1차 함수) : $y = ax + b$
- Linearity (선형성)
 - **Additivity** : $f(x + y) = f(x) + f(y)$
 - **Homogeneity** : $f(ax) = af(x)$

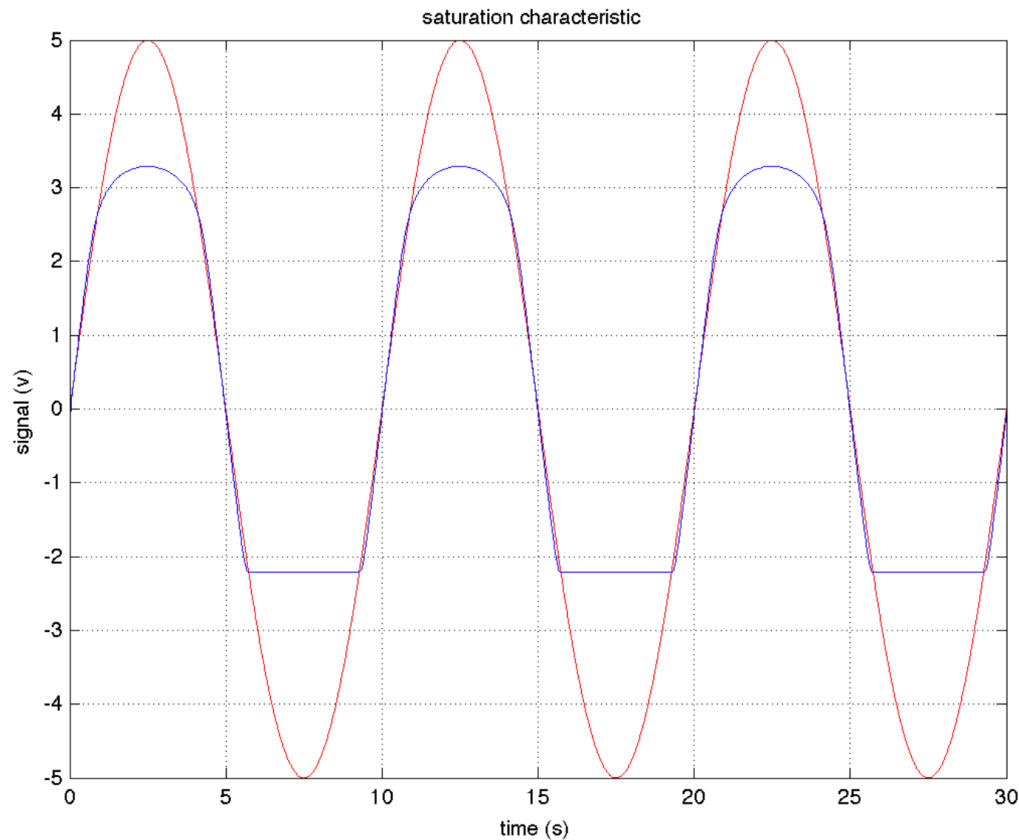
- Example

- $f(x) = \sin x$
- $f(x) = x^2 + 2x + 9$
- $f(x) = -0.87x + 1.12$



Continuous-Time System

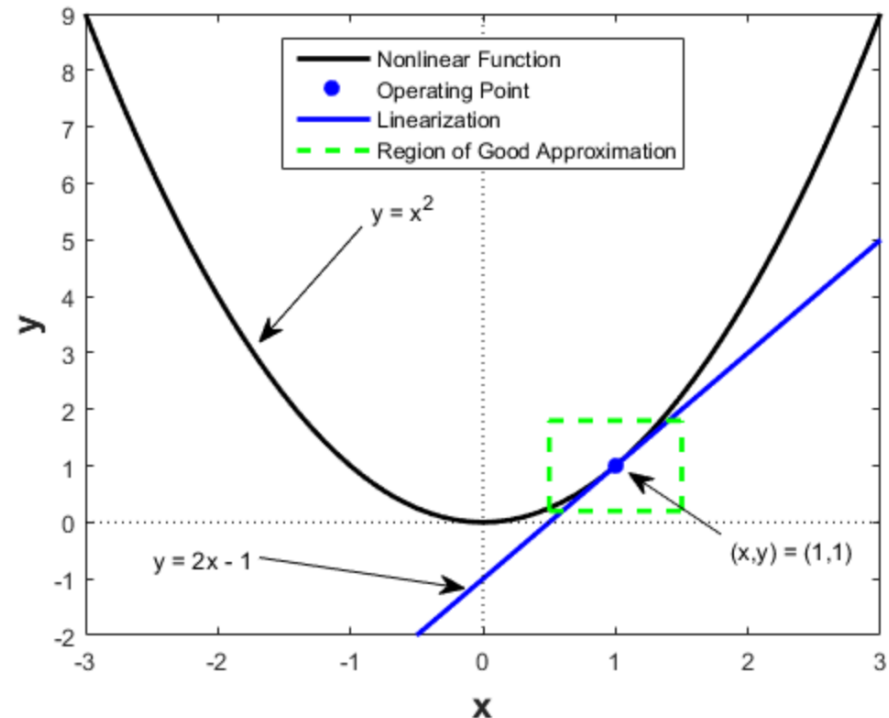
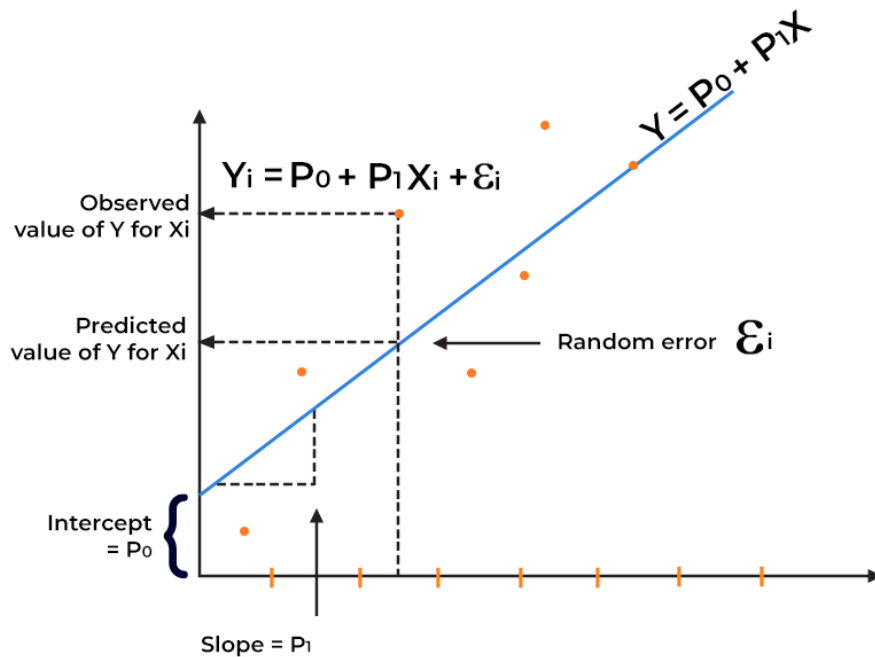
- Linear and Nonlinear System
 - Example: Saturation and Nonlinearity



Continuous-Time System

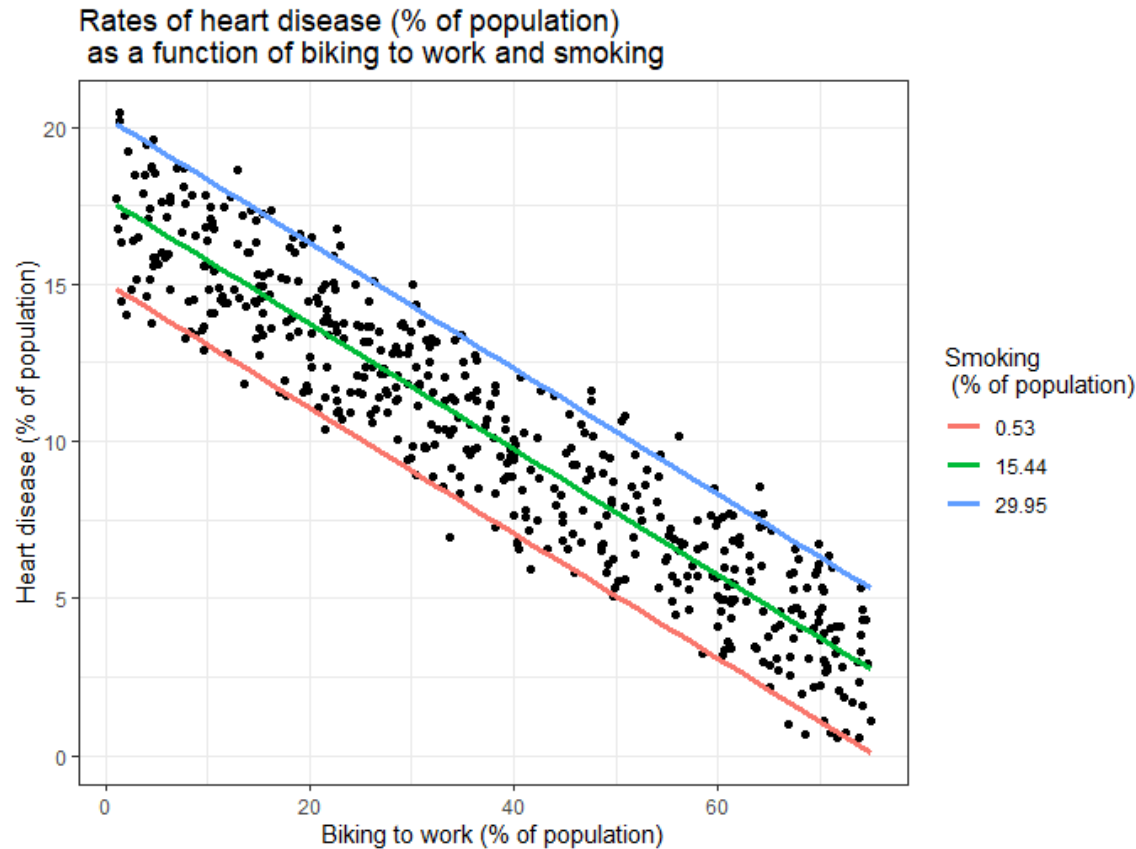
- Nonlinear System

- Regression (회귀) : Linear 한 함수 들 중 가장 적합한 함수를 찾는 법
- Linearization (선형화) : Linear에 가까운 구간만 대체하는 방법



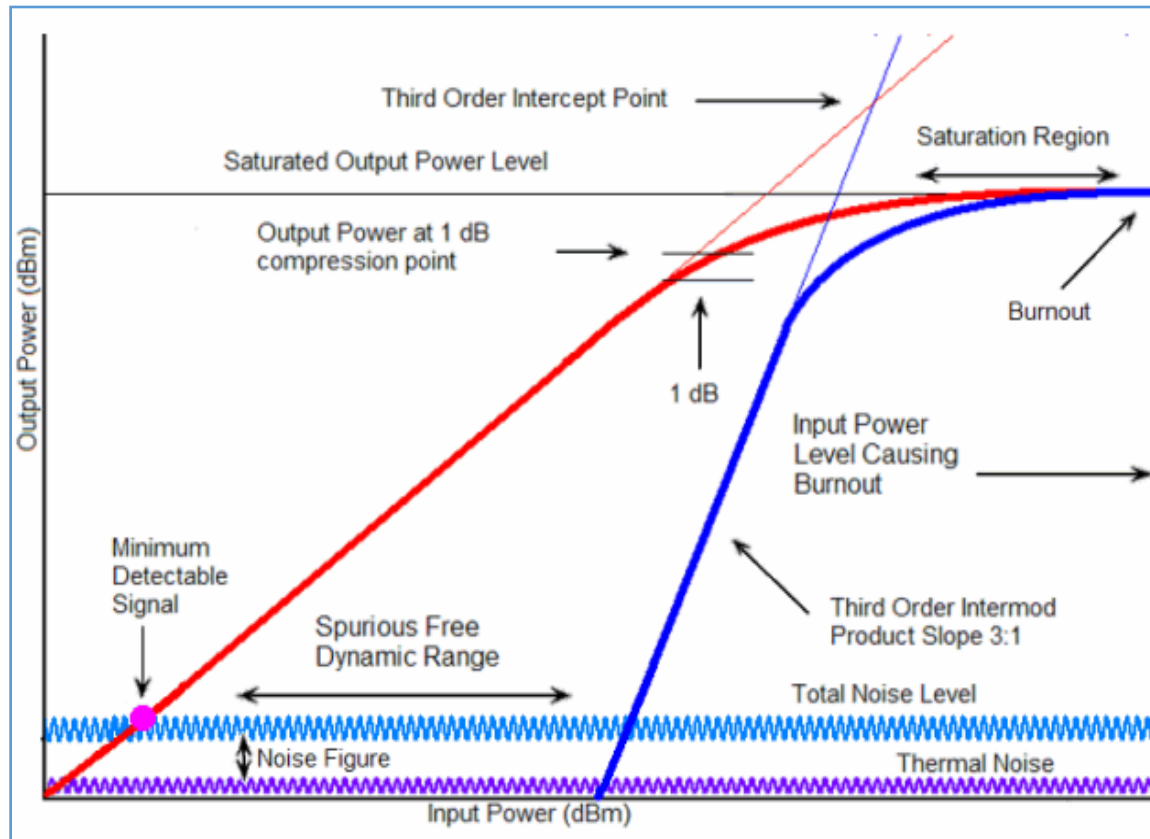
Continuous-Time System

- Nonlinear System
 - Regression : To identify relationships between variables



Continuous-Time System

- Nonlinear System
 - Linearization (선형화) : Linear에 가까운 구간만 사용하는 방법

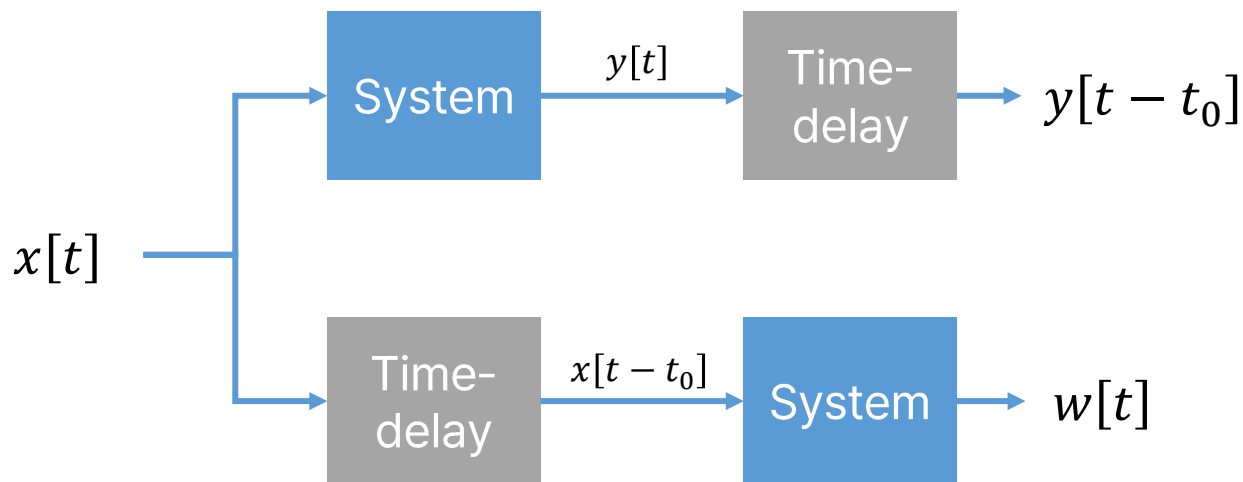


Continuous-Time System

- Time-invariant System

- Time-invariant:** system does not change by time.
- $x[t] \rightarrow y[t]$ 일 때,

$$x[t - \tau] \rightarrow y[t - \tau]$$



$$y[t - t_0] = w[t] \text{ for Time-invariant System}$$

Continuous-Time System

- Time-Invariant System

- Q1. $y(t) = \cos(x(t))$

- Q2. $y(t) = x(t) \cdot \cos t$

Continuous-Time System

- 선형대수학과의 연관성
 - Signal: a **vector** (or a matrix) (*Vector: a matrix with single column/row*)
 - System: a **function** of signal $\rightarrow$ maybe.. calculation of a matrix?
- Example)** $y[n] = x[n] - x[n - 1]$

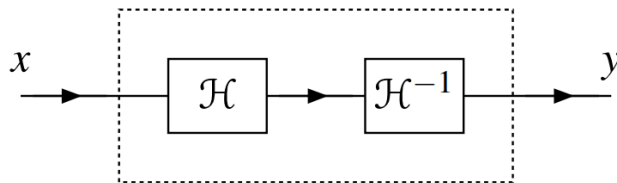
Input Signal	A System $\mathcal{H}$	Output Signal
$N \times 1$	$N \times N$	$N \times 1$
$\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \\ \vdots \\ x[n-1] \end{bmatrix}$	$?$	$\begin{bmatrix} y[0] \\ y[1] \\ y[2] \\ y[3] \\ \vdots \\ y[n-1] \end{bmatrix}$

$y[0] = x[0]$
 $y[1] = x[1] - x[0]$
 $y[2] = x[2] - x[1]$
 $y[3] = x[3] - x[2]$
 $\dots$

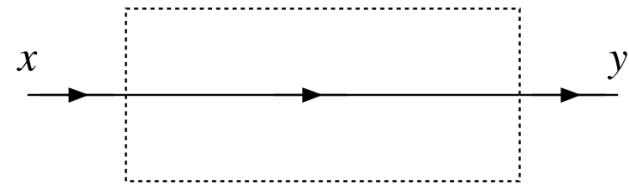
Continuous-Time System

- Invertibility

A system $\mathcal{H}^{-1}$ being the inverse of $\mathcal{H}$ means that the following two systems are equivalent (i.e., $\mathcal{H}^{-1}\mathcal{H}$ is an identity):



System 1: $y = \mathcal{H}^{-1}\mathcal{H}x$



System 2: $y = x$

- How do you know if an inverse matrix exists?

$$\det(A) \neq 0$$

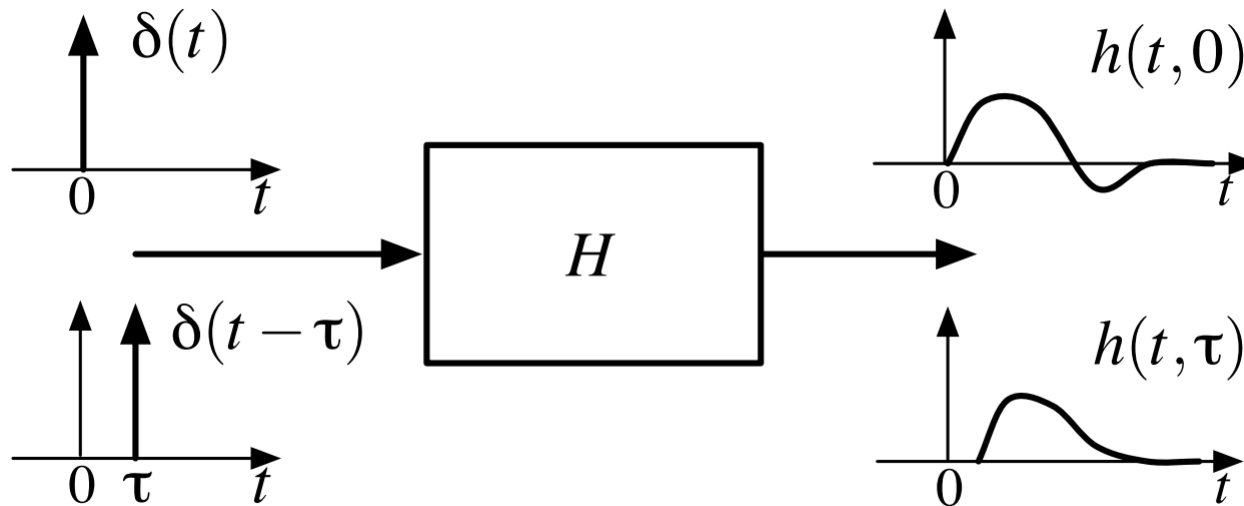
$$X = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Continuous-Time System

- Impulse Response

- An output of the system to an impulse.
- Two time variables : **output time t , impulse time τ**
- If time-invariant system,

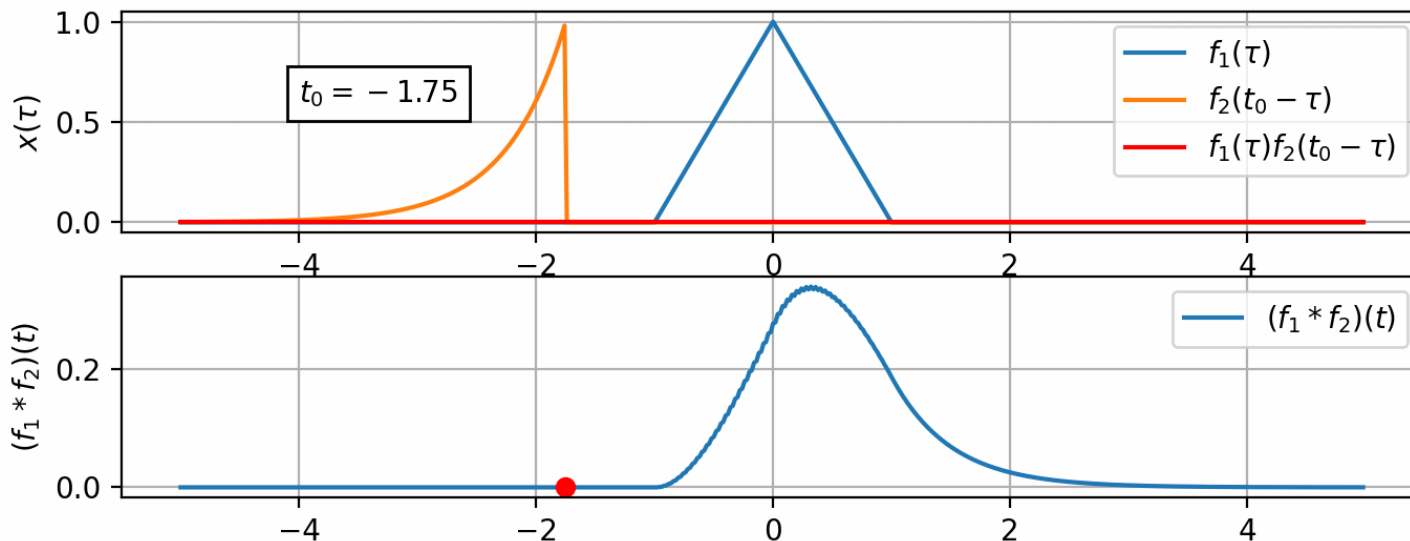
$$h_{\tau}(t) = H(\delta_{\tau}(t))$$



Continuous-Time System

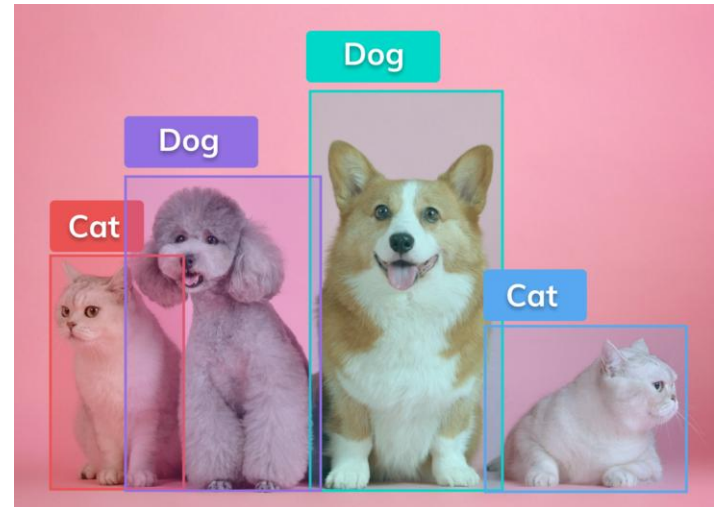
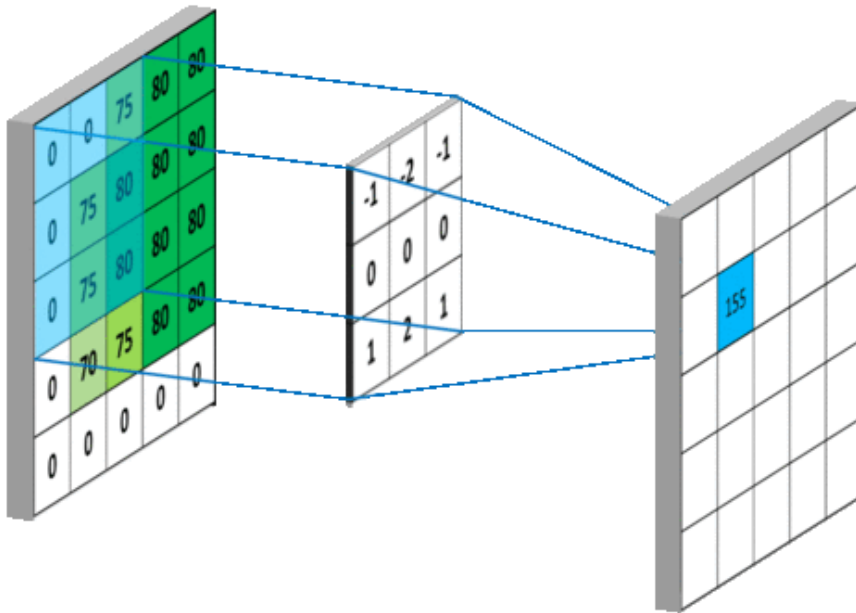
- Convolution
 - A significant operation for **LTI** systems.
 - Note that the function is time-invariant

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau.$$



Continuous-Time System

- Convolution
 - Can be extended in higher dimensional signal.
 - Example) Convolutional Neural Network (CNN)



Continuous-Time System

- Properties of Convolution (*Why?*)

- Commutative

$$x * h = h * x.$$

- Associative

$$(x * h_1) * h_2 = x * (h_1 * h_2).$$

- Distributive

$$x * (h_1 + h_2) = x * h_1 + x * h_2.$$

- Identity

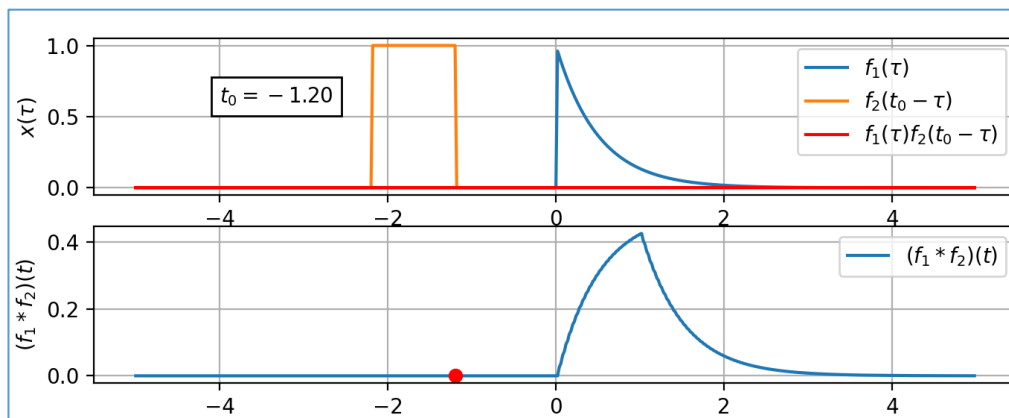
$$x * \delta = x.$$

$$x * \delta(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) d\tau = x(t).$$

Continuous-Time System

- Properties of Convolution (*Why?*)
 - The area under curve

$$\begin{aligned}\int_{-\infty}^{\infty} y(t) dt &= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h(t - \tau) dt \right] d\tau \\ &\quad \text{Area under } h(t) \\ &= \int_{-\infty}^{\infty} x(\tau) [\text{area under } h(t)] d\tau \\ &= \text{area under } x(t) \times \text{area under } h(t)\end{aligned}$$



Continuous-Time System

- Linear, Time-invariant System
 - The missed one : time-invariant

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)d\tau.$$

- The equation becomes simpler,

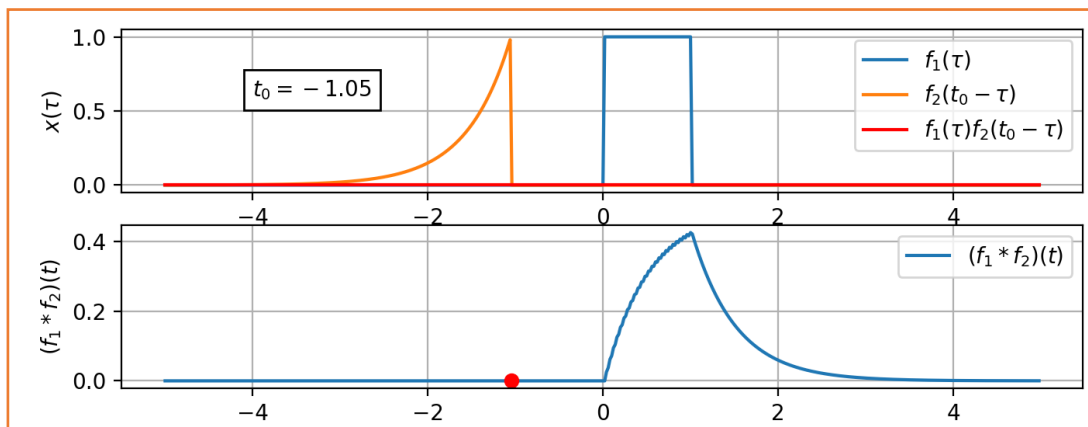
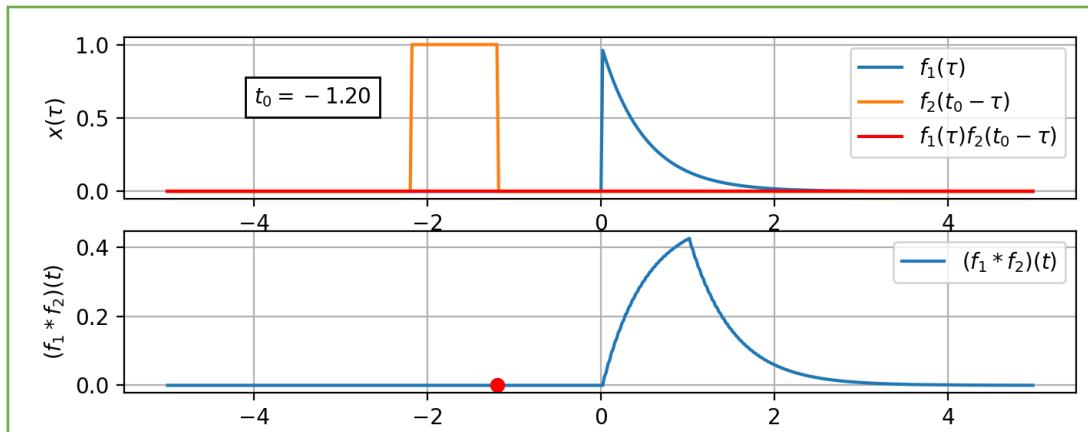
$$y(t) = \int_{-\infty}^{\infty} x(\tau)h_{\tau}(t)d\tau.$$

- "Convolution integral"

Continuous-Time System

- Example 1

- $x(t) = \text{Rect}(t - 0.5), h(t) = e^{-2t} \cdot u(t)$



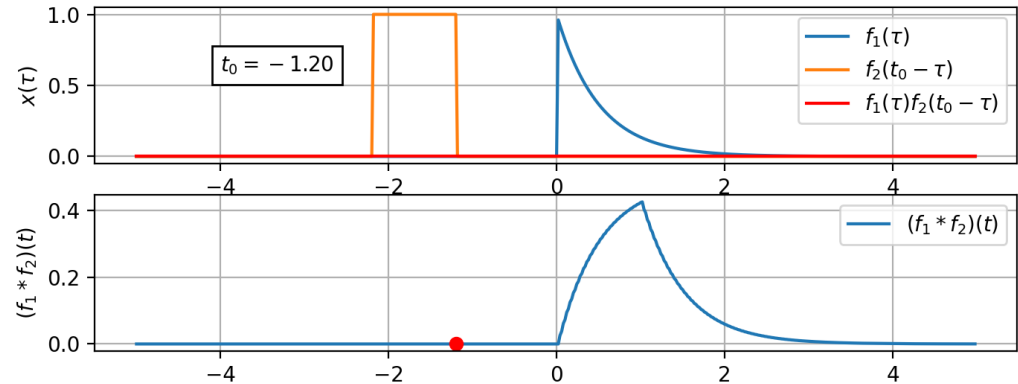
Continuous-Time System

- Example 1

- $x(t) = \text{Rect}(t - 0.5)$
- $h(t) = e^{-2t} \cdot u(t)$

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau.$$

1) For time-shift $\tau < 0$

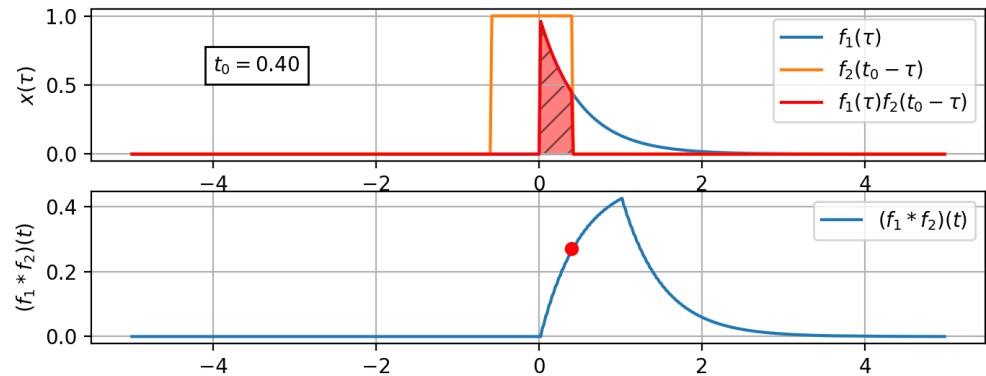


Continuous-Time System

- Example 1

- $x(t) = \text{Rect}(t - 0.5)$
- $h(t) = e^{-2t} \cdot u(t)$

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau.$$



2) For time-shift $0 \leq \tau < 1$

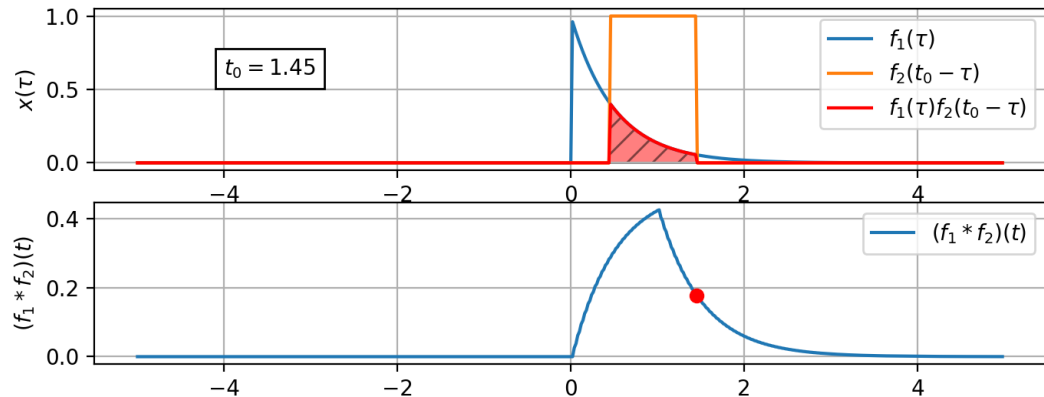
Continuous-Time System

- Example 1

- $x(t) = \text{Rect}(t - 0.5)$
- $h(t) = e^{-2t} \cdot u(t)$

$$x * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau.$$

3) For time-shift $\tau \geq 1$

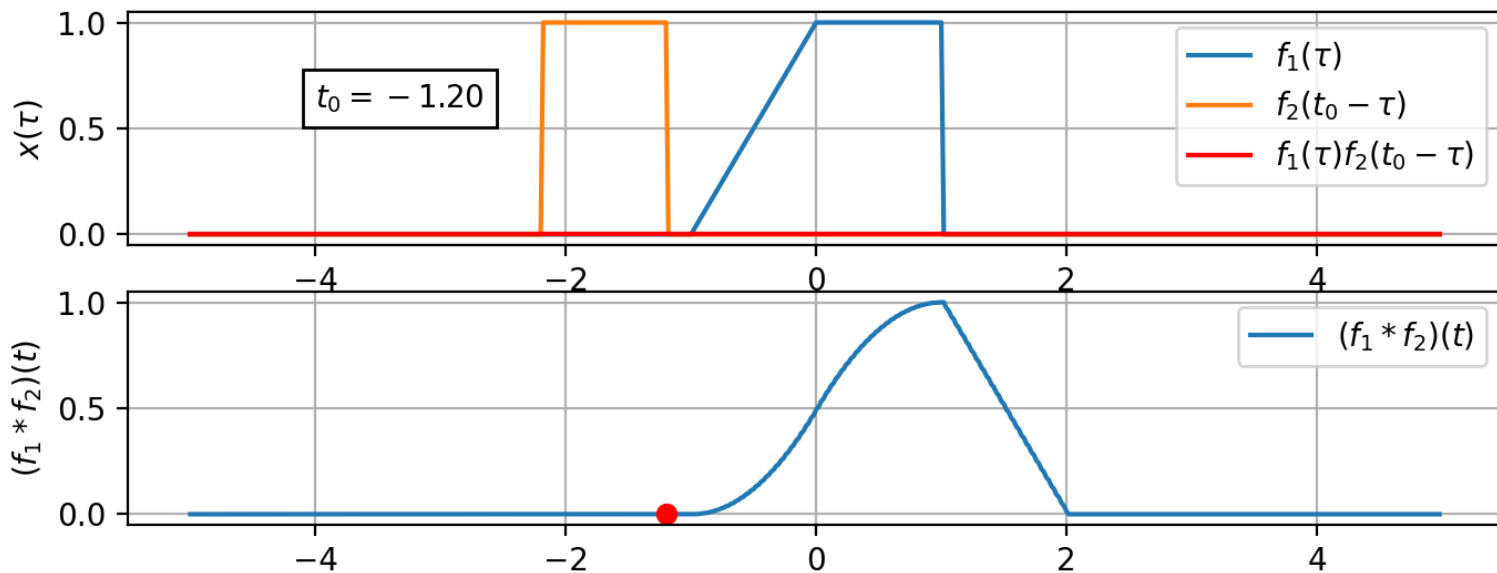


Continuous-Time System

- Example 2

- $x(t) = \text{Rect}(t - 0.5)$

- $$h(t) = \begin{cases} t - 1 & -1 \leq t < 0 \\ 1 & 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

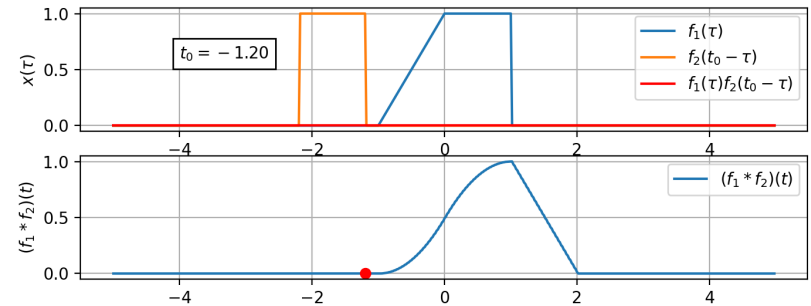


Continuous-Time System

- Example 2

- $x(t) = \text{Rect}(t - 0.5)$

- $$h(t) = \begin{cases} t - 1 & -1 \leq t < 0 \\ 1 & 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$



Continuous-Time System

- Linear, Time-invariant System

signal $x(t) \Leftrightarrow$ **Weighted** Impulses

$$x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(t - \tau) d\tau$$



Apply the system to the input

$$\begin{aligned} y(t) &= H(x(t)) \\ &= H\left(\int_{-\infty}^{\infty} x(\tau) \delta_{\tau}(t) d\tau\right) \end{aligned}$$

If system is LTI
the order can be changed

$$y(t) = \int_{-\infty}^{\infty} x(\tau) H(\delta_{\tau}(t)) d\tau.$$

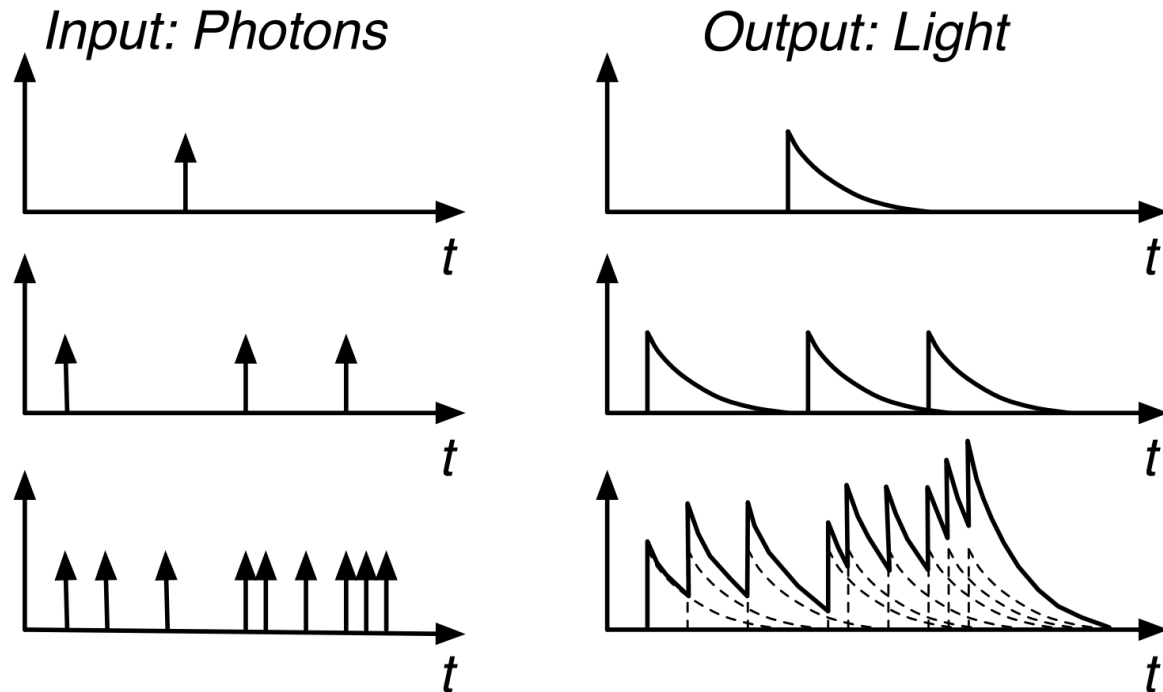
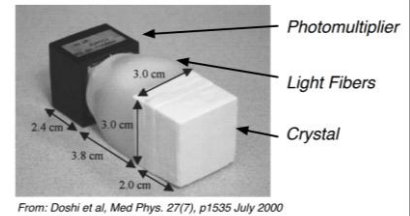


An LTI system is represented by
conv(input, Impulse Resp.)

$$y = x * h.$$

Continuous-Time System

- Linear, Time-invariant System
 - High energy photon detectors
 - Output is superposition of impulse responses

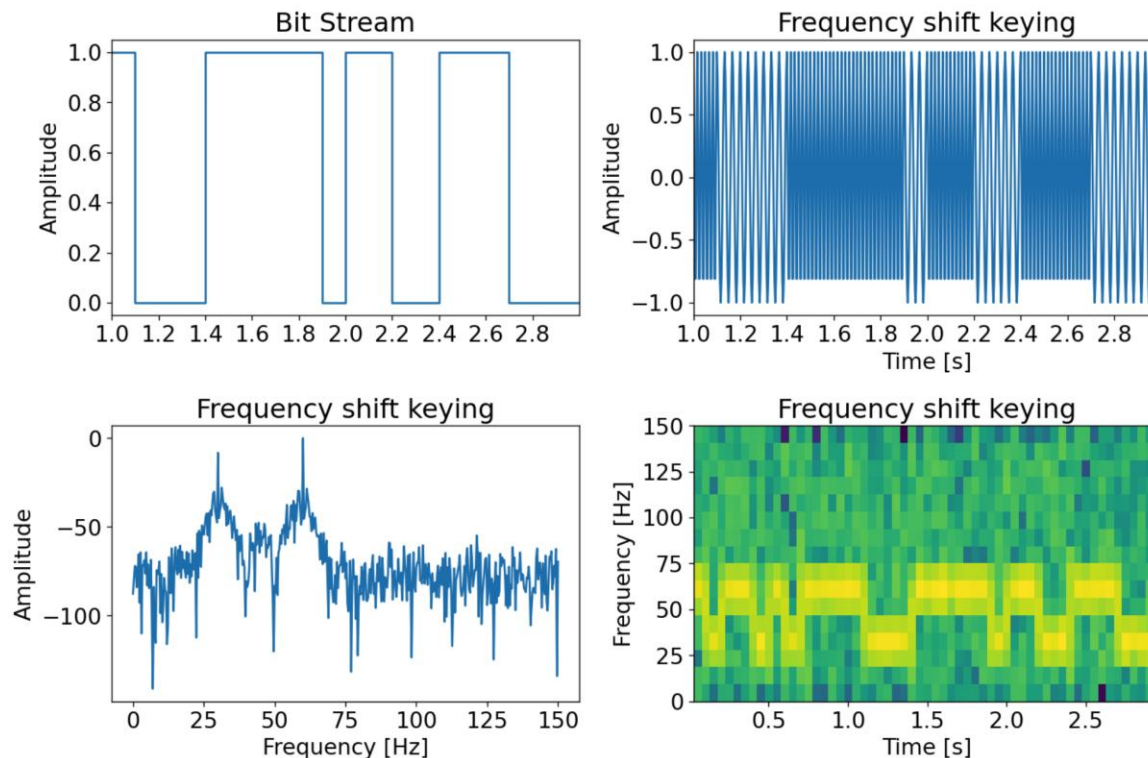


4. Complex Numbers

Why do we need complex numbers?

Complex Numbers

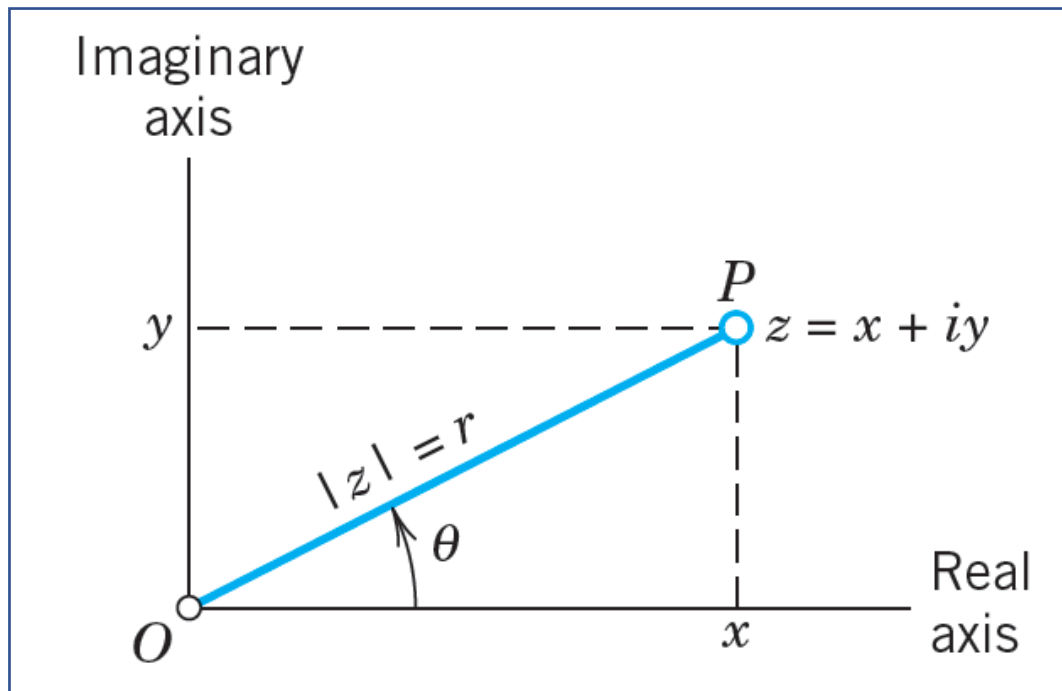
- Complex Calculus and Analysis
 - An electrical signal → **Amplitude, Frequency, Phase**
 - A communication signal : series of **complex numbers**!



Complex Numbers

- A number that describes two number!

$$z = x + iy = re^{i\theta}$$



Complex Numbers

- Complex Conjugate Numbers
 - Notation

$$z = x + iy \leftrightarrow \bar{z} = x - iy$$

- Characteristic → To calculate power of electric field, ...

$$z\bar{z} = x^2 + y^2$$

$$\operatorname{Re} z = x = \frac{1}{2}(z + \bar{z})$$

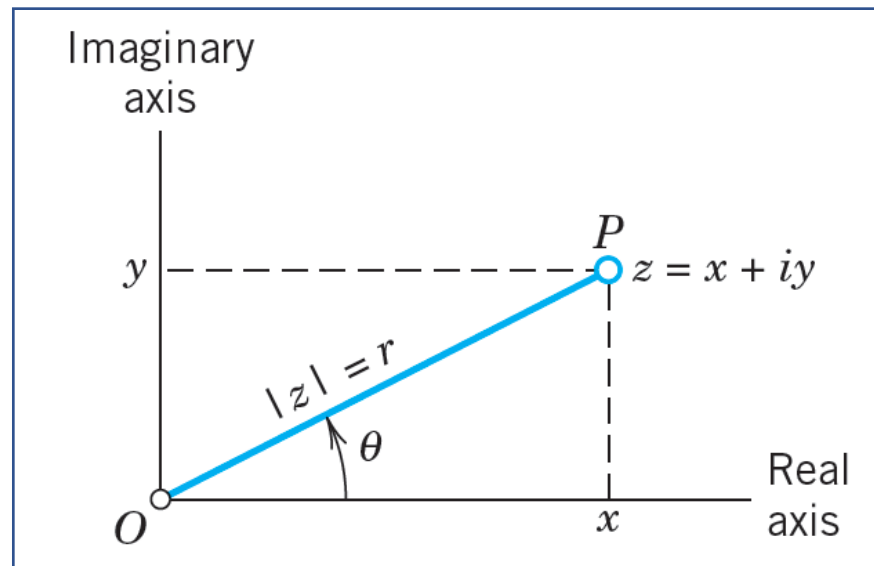
$$\operatorname{Im} z = y = \frac{1}{2i}(z - \bar{z})$$

Complex Numbers

- The **Polar** Coordinate

$$z = x + iy$$

$$z = re^{i\theta}$$



Complex Numbers

- Multiplication and Division in Polar Form

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1) \text{ and } z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$$

$$\begin{aligned} z_1 z_2 &= r_1 r_2 (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) \\ &\quad + i (\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2) \end{aligned}$$

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$$

$$|z_1 z_2| = |z_1| |z_2|$$

By letting $z_1 = (z_1/z_2)z_2$, Division of polar form can be obtained.

Complex Numbers

- Multiplication and Division in Polar Form
 - In summary,

	Multiplication	Division
Absolute Value (Modulus)	$ z_1 z_2 = z_1 z_2 $	$\left \frac{z_1}{z_2} \right = \frac{ z_1 }{ z_2 }$
Argument	$\arg(z_1 z_2) = \arg z_1 + \arg z_2$	$\arg(z_1 / z_2) = \arg z_1 - \arg z_2$

- Examples

Complex Numbers

- Roots of Complex number

$$z = w^n \leftrightarrow w = \sqrt[n]{z}$$

– Let

$$z = r(\cos \theta + i \sin \theta) \quad \text{and} \quad w = R(\cos \phi + i \sin \phi)$$
$$w^n = R^n(\cos n\phi + i \sin n\phi) = z = r(\cos \theta + i \sin \theta)$$

- Comparing Absolute value and Arguments,

$$R^n = r, \quad n\phi = \theta + 2k\pi \left(\phi = \frac{\theta}{n} + \frac{2k\pi}{n} \right)$$

Complex Numbers

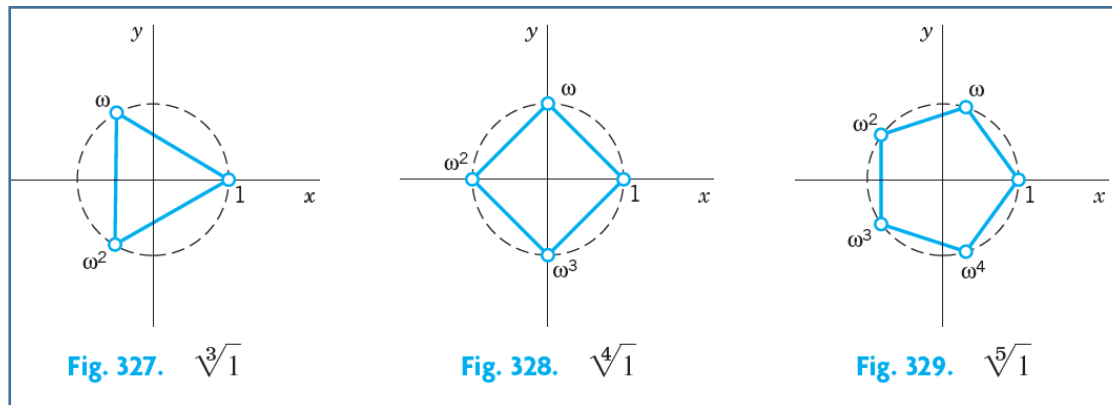
- Roots of Complex number

- In shortly,

$$\sqrt[n]{z} = \sqrt[n]{r} \left(\cos \frac{\theta + 2k\pi}{n} + i \sin \frac{\theta + 2k\pi}{n} \right)$$

- And in case of $z = r = 1, (\arg z = 0)$

$$\sqrt[n]{1} = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n}$$



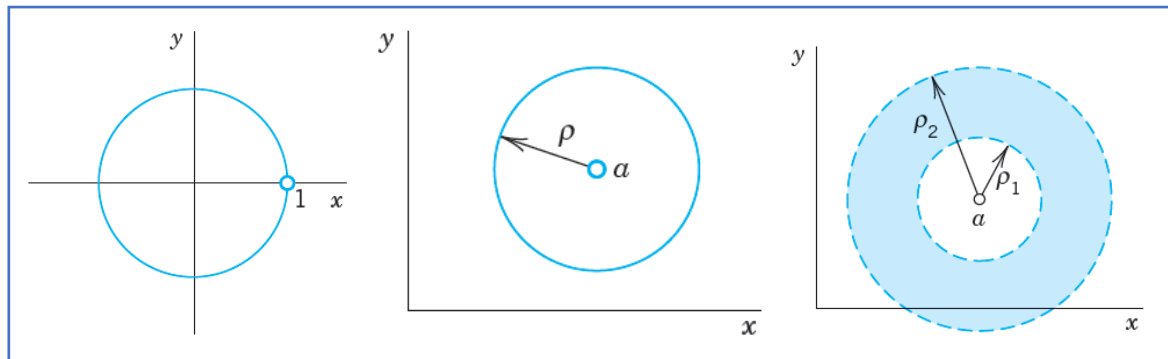
Complex Numbers

- Functions in a complex plane

$$w = f(z) = u(x, y) + iv(x, y)$$

- w is value of f at z .
- z is called a complex variable.
- Assume to be **single-valued** function

- Circles and Disk in a complex plane $|z - a| = \rho$



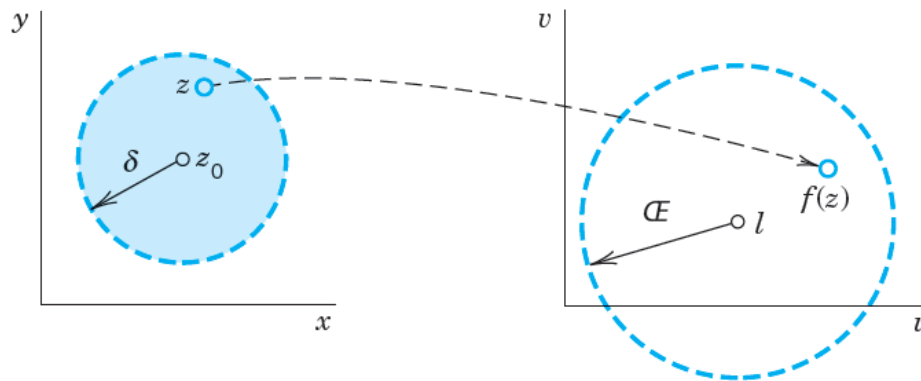
Complex Numbers

- Limit and Continuity in Complex Plane

$$\lim_{z \rightarrow z_0} f(z) = l$$

– A function $f(z)$ is continuous at $z = z_0$,

If $f(z_0)$ is defined and $\lim_{z \rightarrow z_0} f(z) = f(z_0)$



Complex Numbers

- Derivative in Complex Plane

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

① Differentiability at z_0

② If $f(z)$ is differentiable at z_0 , then it is also continuous

– Differentiation rules

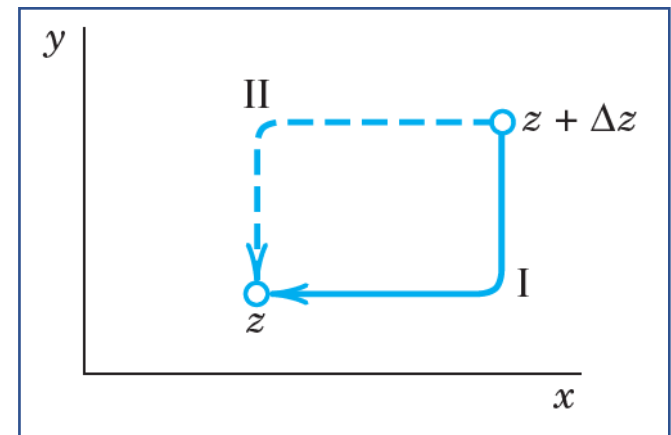
$$(cf)' = cf', \quad (f + g)' = f' + g', \quad (fg)' = f'g + fg', \quad \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Complex Numbers

- Ex 4. Not differentiable!

$$f(z) = \bar{z} = x - iy$$

- if $\Delta y = 0$ (path I)
- if $\Delta x = 0$ (path II)



Complex Numbers

- Expanding some functions into complex domain.

$$e^z \text{ or } \exp z = e^x (\cos y + i \sin y)$$

- (A) $e^z = e^x$ for real $z = x$ because $\cos y = 1$ and $\sin y = 0$ when $y = 0$.
- (B) e^z is analytic for all z .
- (C) The derivative of e^z is e^z .

- For $z = iy$, the Euler formula is obtained

$$e^{iy} = \cos y + i \sin y$$

Complex Numbers

- Real Trigonometric Function to Complex

$$e^{ix} = \cos x + i \sin x, e^{-ix} = \cos x - i \sin x$$

$$\cos z = \frac{1}{2} (e^{iz} + e^{-iz}), \quad \sin z = \frac{1}{2i} (e^{iz} - e^{-iz})$$

- Note that Euler's formula is valid in complex number z

$$e^{iz} = \cos z + i \sin z$$

Complex Numbers

- Complex Hyperbolic Functions and Relationships

$$\cosh z = \frac{1}{2} (e^z + e^{-z}), \quad \sinh z = \frac{1}{2} (e^z - e^{-z})$$

$$\cosh iz = \cos z, \quad \sinh iz = i \sin z$$

$$\cos iz = \cosh z, \quad \sin iz = i \sinh z$$

Complex Numbers

- Complex logarithm

$$w = \ln z \leftrightarrow e^w = z$$

- Setting $w = u + iv$ and $z = re^{i\theta}$,

$$e^w = e^{u+iv} = re^{i\theta} \rightarrow e^u = r, \quad v = \theta$$

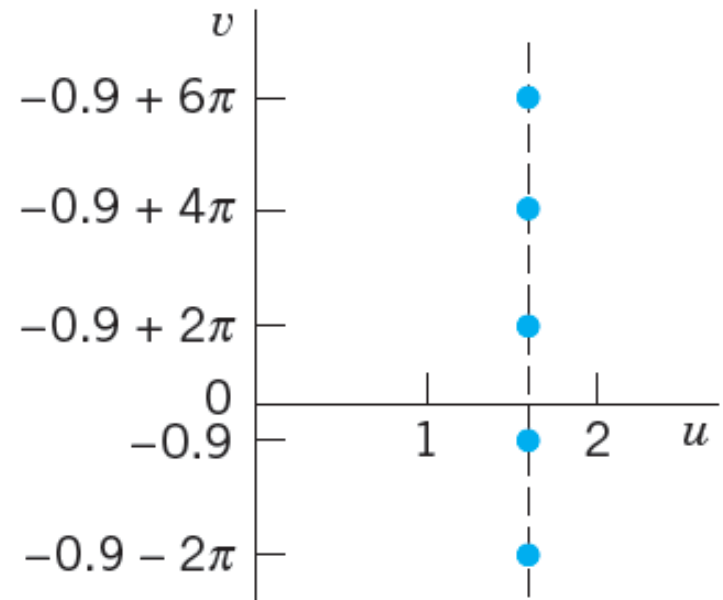
$$\ln z = \ln r + i\theta$$

- The arg. of z is determined up to integer multiples of 2π
- Principal value of $\ln z$: $\text{Ln } z = \ln z + i \arg z$
- What is $\ln i$?

Complex Numbers

$$\ln z = \ln r + i\theta$$

- Ex 1. Principal Value of Complex Natural Logarithm
 - $\ln(3 - 4i) = \ln 5 + i \arg(3 - 4i) = 1.61 - 0.93 \pm 2n\pi i$
 - $\text{Ln}(3 - 4i) = 1.61 - 0.93i$



Complex Numbers

- Definition of Analyticity (Holomorphic)

Analyticity

A function $f(z)$ is said to be *analytic in a domain D* if $f(z)$ is defined and differentiable at all points of D . The function $f(z)$ is said to be *analytic at a point $z = z_0$* in D if $f(z)$ is analytic in a neighborhood of z_0 .

Also, by an **analytic function** we mean a function that is analytic in *some* domain.

Complex Numbers

- Most important equation in this chapter
 - Cauchy-Riemann Equation
 - Is this function analytic in **domain D**?

$$u_x = v_y \quad u_y = -v_x$$
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Cauchy–Riemann Equations

Let $f(z) = u(x, y) + iv(x, y)$ be defined and continuous in some neighborhood of a point $z = x + iy$ and differentiable at z itself. Then, at that point, the first-order partial derivatives of u and v exist and satisfy the Cauchy–Riemann equations (1).

Hence, if $f(z)$ is analytic in a domain D , those partial derivatives exist and satisfy (1) at all points of D .

Complex Numbers

- Problem Set 13.4 Q.2

$$f(z) = iz\bar{z}$$

Complex Numbers

- Is logarithm analytic?

$$\ln z = \text{Ln } z \pm 2n\pi i$$

- Yes, for limited region that z is not 0 or negative real.

$$(\ln z)' = \frac{1}{z}$$

Complex Numbers

- General Power of Complex number $z = x + iy$

$$z^c = e^{c \ln z}, a^z = e^{z \ln a}$$

- The principal value of $z^c = e^{c \operatorname{Ln} z}$

- Example 3. General Power i^i

$$i^i = e^{i \ln i} = \exp(i \ln i) = e^{-(\pi/2) \mp 2n\pi}$$

$$\left(\ln i = 0 + i \arg i = \frac{\pi i}{2}, -\frac{3\pi i}{2}, \dots \right)$$

Complex Numbers

- A number that describes two number!

$$z = x + iy = re^{i\theta}$$

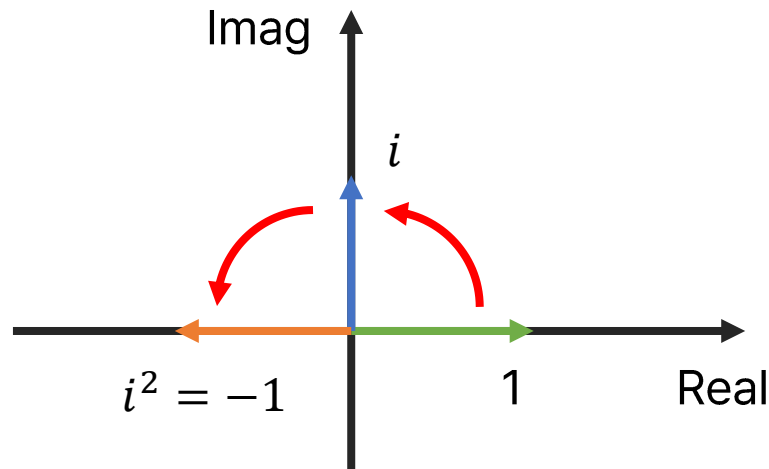
- Addition & Subtraction => Cartesian form
- Multiplication & Division => Polar form (a.k.a. Phasor)

	Multiplication	Division
Absolute Value (Modulus)	$ z_1 z_2 = z_1 z_2 $	$\left \frac{z_1}{z_2} \right = \frac{ z_1 }{ z_2 }$
Argument	$\arg(z_1 z_2) = \arg z_1 + \arg z_2$	$\arg(z_1 / z_2) = \arg z_1 - \arg z_2$

Complex Numbers

- Euler's Identity : Rotation in complex plane.

$$e^{i\pi} = \left(e^{i\frac{\pi}{2}}\right)^2 = i^2 = -1$$

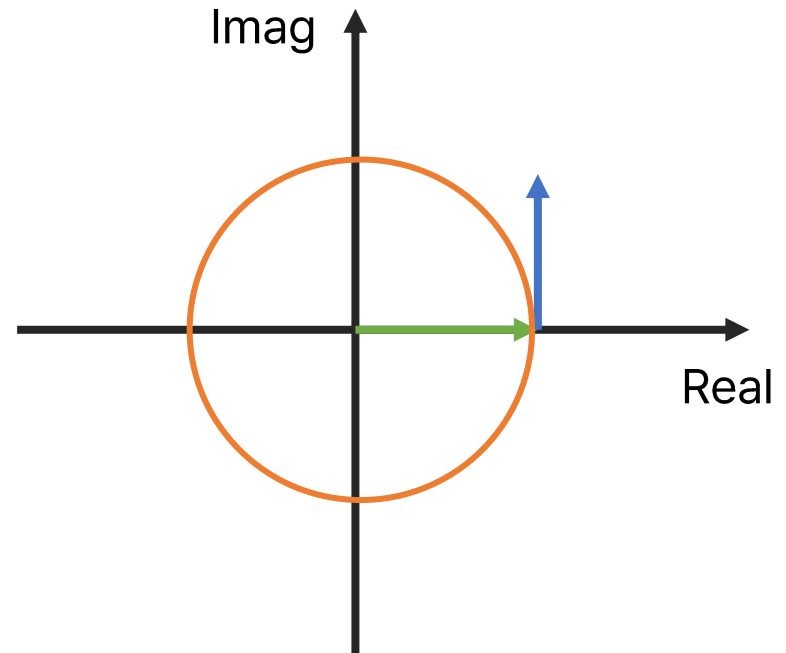


Complex Numbers

- Euler's Identity : Derivative Equation. (미분 방정식)

- Consider $\frac{d}{dt} e^{it} = i e^{it}$
(위치) (속도)

- Multiplication of i
=> Rotation by $\pi/2$



Complex Numbers

- Euler's Identity : Taylor Series

$$e^{i\pi} + 1 = 0$$

- Substituting ix ,

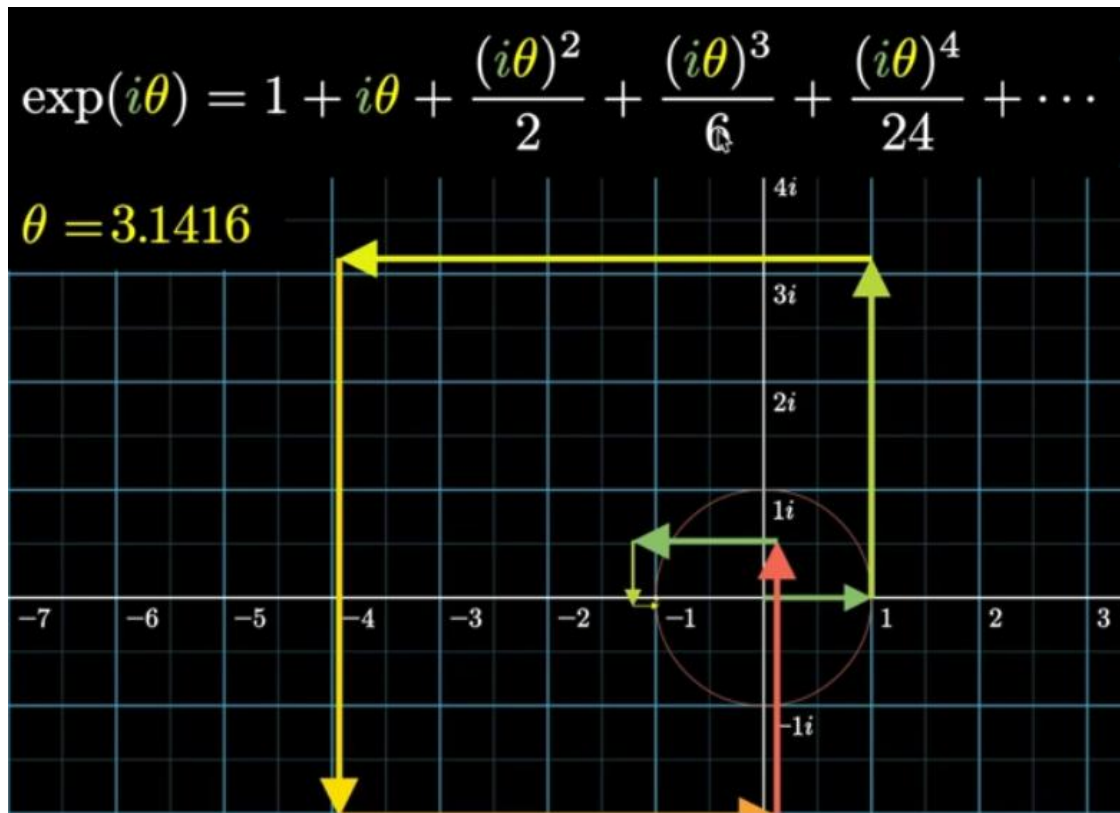
$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \quad \text{for all } x$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \quad \text{for all } x$$

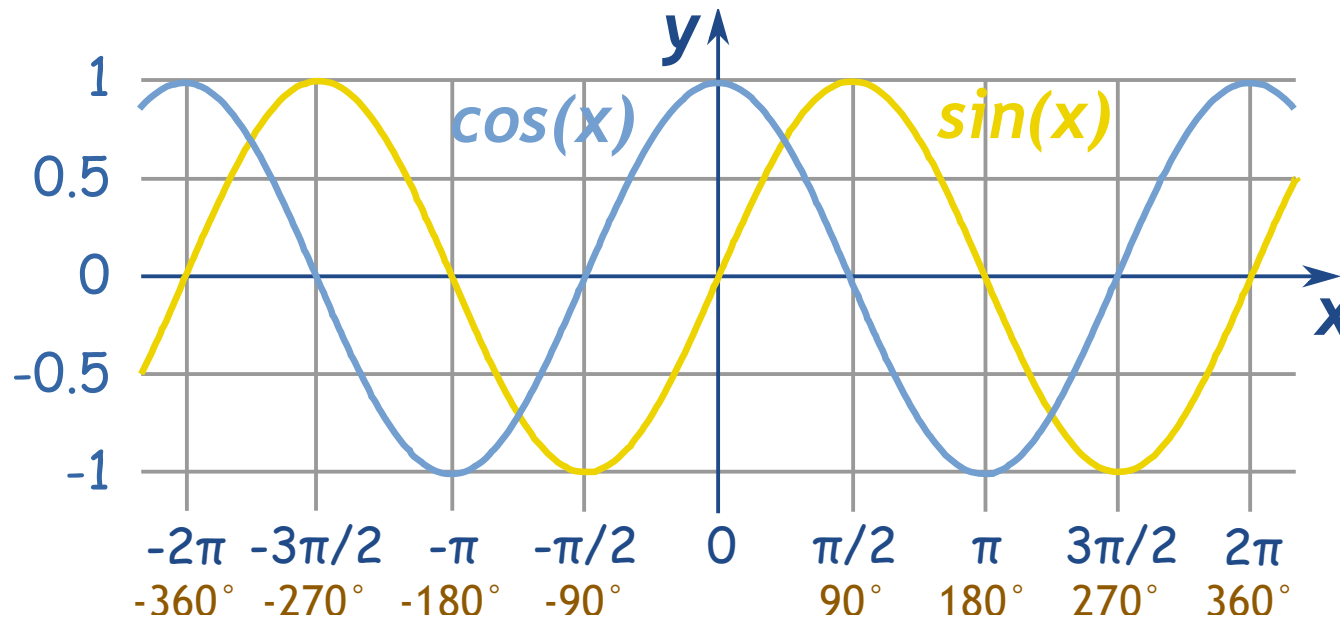
Complex Numbers

- Euler's Identity : Taylor Series in Complex Plane
 - **The series == Vector Addition**



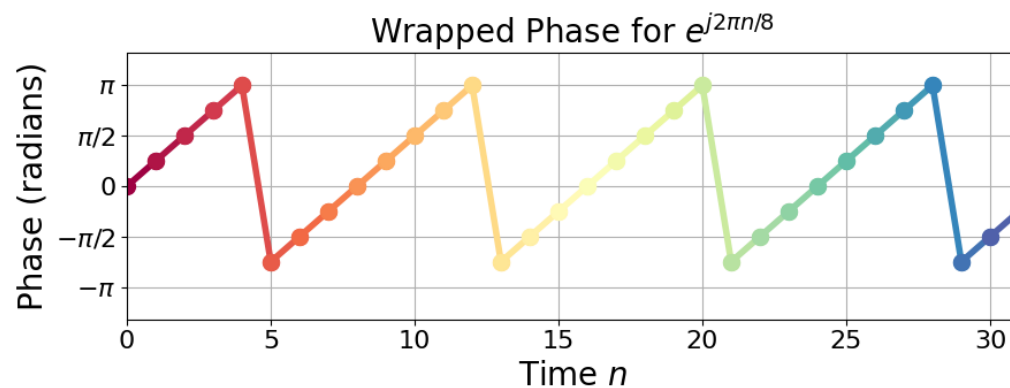
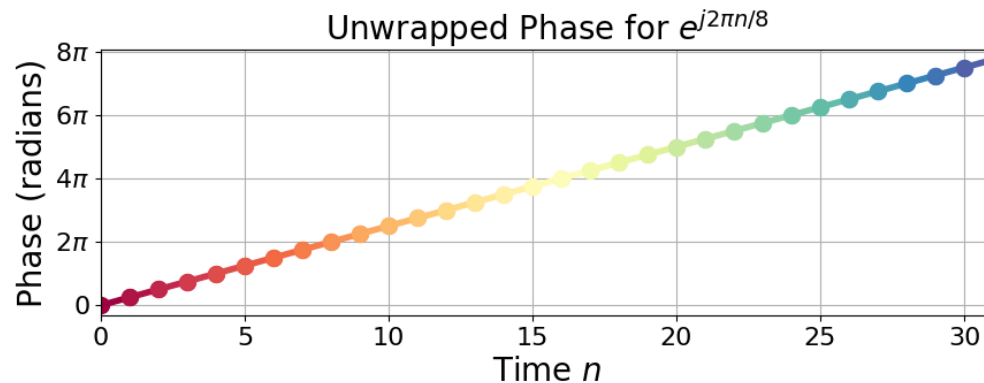
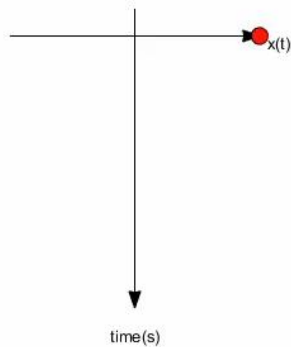
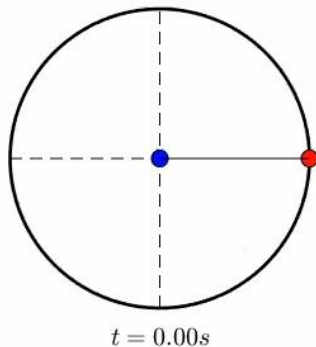
Complex Numbers

- A sinusoidal signal
 - Amplitude [voltage]
 - Phase [degree, radian]
 - Frequency [Hz]



Complex Numbers

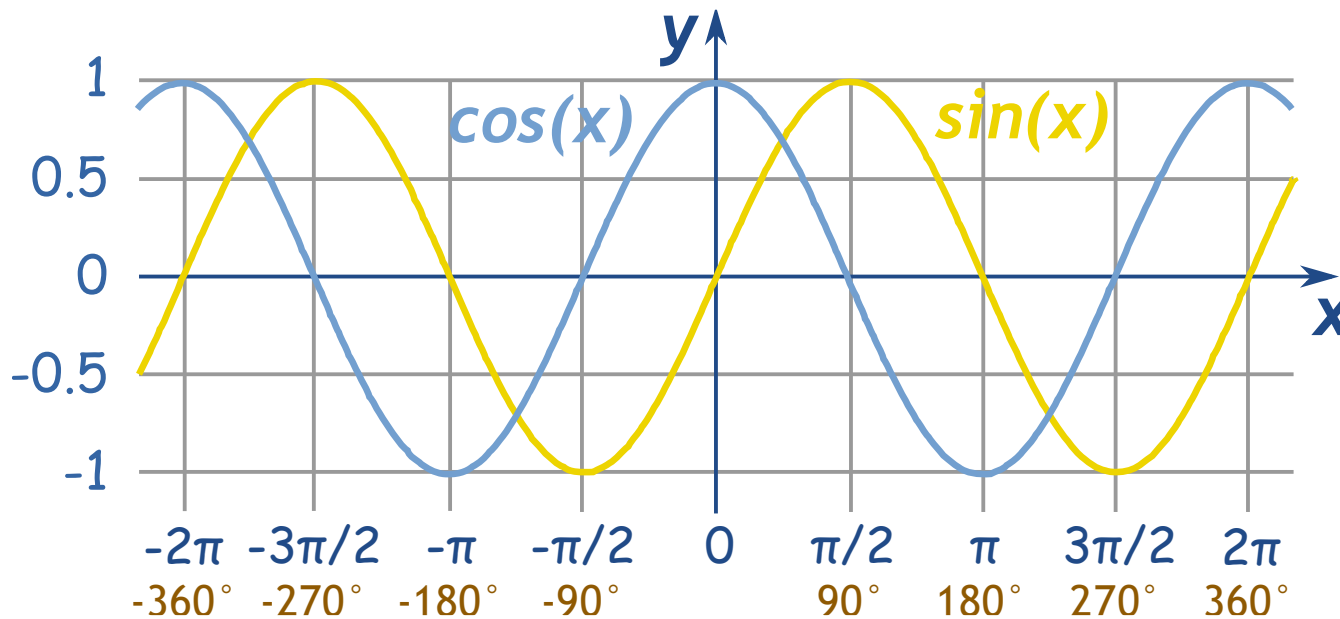
- The phase ambiguity
 - $\mathbb{R} \Rightarrow [0, 2\pi]$



Complex Numbers

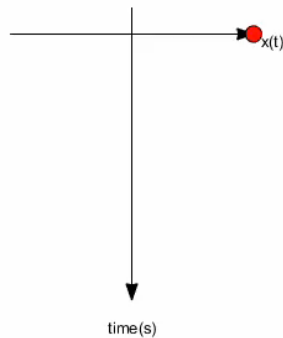
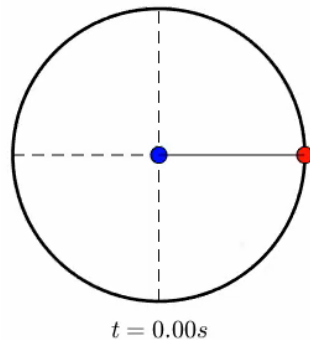
- Think about the **orthogonality**

$$\int \sin(kx) \cdot \cos(nx) dx = 0$$



Complex Numbers

- Euler's Identity
 - Period : A time to one rotation
 - Frequency : The speed of rotation



$$\omega = 2\pi f$$

$$e^{j\omega t} = \cos \omega t + j \sin \omega t$$

Complex Numbers

- Euler's Identity
 - **Complex exponential** is also orthogonal

$$e^{j\omega t} = \cos \omega t + j \sin \omega t$$

$$\int e^{j\omega_1 t} \cdot e^{j\omega_2 t} dt = 0?$$

Thank you

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